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MOTIONS OF A POROUS PARTICLE IN STOKES FLOW: PART 1. UNBOUNDED SINGLE-FLUID DOMAIN PROBLEM

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Abstract—Exact solutions in closed form have been found using the eigenfunction-expansion method for various linear and quadratic flows of an *unbounded* incompressible viscous fluid at low Reynolds number past a *porous* sphere with a uniform permeability distribution. The linear flows considered here are a simple shear and an axisymmetric uniaxial straining flows and the quadratic flows include a unidirectional paraboloidal and a stagnation-like flows as typical representations. The theoretical analysis determines a general motion of a freely suspended particle in the prescribed mean flow at infinity. Then the solutions are expressed in terms of fundamental singularity solutions for Stokes flow which will be applied to examine the motion of a porous sphere in the presence of a plane fluid-fluid interface in the forthcoming part of the present paper.

INTRODUCTION

This paper examines a problem in which a porous particle is immersed in either a linear or quadratic mean flow at infinity in an *unbounded* single-fluid domain. The flow of viscous fluid past a porous particle has been a problem of long-standing interest, corresponding to various types of application. For example, an arbitrary motion of a porous particle through a fluid at rest at infinity is relevant to sedimentation phenomena of flocs and particle clouds [cf. Ooms et al.[1]; Felderhof and Deutch [2]; Felderhof [3]; Matsumoto and Suganuma [4]; among others]. Particle motion in more general flow fields such as pure straining flow is relevant in suspension mechanics, and to some aspects of the modeling of polymer molecules to account for the hydrodynamic interaction between polymer segments [cf. Felderhof [5]; Deutch and Felderhof [6]].

The study of the porous media problem begins with Darcy's law, according to which the average fluid velocity is proportional to the average pressure gradient. Darcy's law has been applied to various problems involving flow through porous media and has proved to be reliable model for creeping flow in the interior of statistically homogeneous isotropic materials. Despite of its success for interior flow, Darcy's law is not a complete model for a porous medium of finite size bordered by regions of pure fluid. Under normal circumstances, we require continuity of velocity and surface stress across the boundary; however, this is not possible due to the reduced order of Darcy's law as compared with the Stokes equation. To circumvent the singular behavior of Darcy's law at the boundary, Beavers and Joseph [7] proposed, on heuristic grounds, a slip boundary condition which allows discontinuity in the fluid velocity across the particle surface. Saffman [8] and others have offered theoretical justification on the slip condition of Beavers and Joseph. A second approach to the problem of matching the interior and exterior flows is to replace Darcy's law with Brinkman equation of the same order as the Stokes equation.

Brinkman equation is based on a prima facie assumption that the resistance due to the solid inclusions is proportional to the local relative velocity of the fluid and solid phases [cf. Brinkman [9]; Debye and Bueche [10]]. It is however made with reliance on physical intuition and the *a posteriori* justification by the success of the hypothesis [cf. Larson and Higdon [11] and references therein]. Howells [12] determined the Green's function (i.e., the fundamental solution for a point force) of Brinkman equation to calculate the drag on an impermeable particle immersed in an isotropic porous medium comprised of a random array of particles, and gave justification of Brinkman's model as providing a first approximation for the mean flow past a sphere in the case of sparse distribution. Kojima [13] employed the Green's functions for Stokes equation and for Brinkman equation as found by Howells to construct integral solutions valid in their respective domains, and outlined the method for calculating the translational friction and the intrinsic viscosity for dilute suspensions of porous spheres by considering asymptotic forms of the integral equations for large or small permeability limit. In another study, Matsumoto and Suganuma [4] measured the settling velocity of model flocs of steel wool and found good agreement with the velocity predicted by Brinkman equation. In practice, Brinkman equation appears more useful, because it incorporates a more fundamental analysis and embodies a well-defined stress tensor.

In the present effort we examine the case of a porous sphere immersed in a Newtonian fluid in an *unbounded* domain which is undergoing a linear or a quadratic mean flow at infinity. The method of solution employed here is the eigenfunction-expansion for both Stokes and Brinkman equations in spherical coordinate system. The linear flows considered here are a simple shear and an axisymmetric straining flows and the quadratic flows include a unidirectional paraboloidal and a stagnation-like flows. The theoretical analysis determines a general motion of a freely suspended particle in the prescribed mean flow. Then the solutions are expressed in terms of fundamental singularity solutions for Stokes flow to investigate the motion of porous particles in the presence of a plane fluid-fluid interface in the forthcoming part of this work. The solution scheme for the latter problem is the reflections-method in conjunction with the corresponding solutions for an unbounded single-fluid domain [cf. Lee et al. [14]; Yang and Leal [15]].

BASIC EQUATIONS

We begin by considering the governing equations and boundary conditions for the flow fields both interior and exterior to a porous sphere immersed in an incompressible Newtonian fluid which, at large distance from the particle, is undergoing an undisturbed flow defined by a velocity $\mathbf{U}^\infty(\mathbf{x})$ and pressure $P^\infty(\mathbf{x})$. The variables in this analysis may be considered to be nondimensionalized with respect to the characteristic variables: u_c (velocity), $l_c= a$(length, i.e., sphere radius a) and $p_c=\frac{\mu u_c}{a}$(stress). We take a coordinate system in which the particle is placed at origin, $\mathbf{x}=(x, y, z)$ being the position vector, and $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ the base vectors in the x, y, z directions, respectively. It is assumed that the Reynolds number for the motion is sufficiently small that the quasi-steady approximation is applicable. The equations of motion therefore reduce to steady Stokes equation plus continuity equation for the flow outside the particle, i.e.,

$$\nabla \cdot \tau^o = -\nabla p^o + \nabla^2 \mathbf{u}^o = 0 \tag{1a}$$

$$\nabla \cdot \mathbf{u}^o = 0 \tag{1b}$$

where the superscript "o" denotes the flow field exterior to the porous sphere.

The flow of a viscous fluid through an isotropic porous medium can be analyzed by studying a simple continuum model, consisting of a random array of solid objects which is permeable to the flow and exerts a friction proportional to the local relative velocity of the fluid and the solid phases:

$$\nabla \cdot \tau^i = -\nabla p^i + \nabla^2 \mathbf{u}^i = \frac{a^2}{k}\mathbf{u}^i \tag{2a}$$

$$\nabla \cdot \mathbf{u}^i = 0 \tag{2b}$$

in which the superscript "i" represents the flow field interior to the sphere. The Brinkman equation, (2a), proposed independently by Brinkman [9] and Debye and Bueche [10], is based on the creeping motion equations for low Reynolds number flow with an additional friction $\frac{a^2}{\kappa}\mathbf{u}^i$ to account for the local resistance arising from the discontinuities in shear stress and pressure across the solid phases. We can identify $\frac{a^2}{\kappa}\mathbf{u}^i$ as the force per unit volume that the fluid exerts upon the solid surface beyond the hydrostatic force and beyond any force attributable to the ambient pressure. When the viscous term is omitted in (2a), the Brinkman equation reduces to the Darcy's law for a porous medium with the permeability κ. Although (2a) was originally derived from heuristic arguments, it has since received theoretical and experimental justification from numerous authors [e.g., Howells [12]; Felderhof and Deutch [3]; Matsumoto and Suganuma [4]; Larson and Higdon [11] and others].

The boundary conditions for the equations, (1) and (2), are

$$\mathbf{u}^o \to \mathbf{U}^\infty(\mathbf{x}), \quad p^o \to P^\infty(\mathbf{x}) \quad as \quad r=|\mathbf{x}| \to \infty \tag{3a}$$

plus the continuity of velocity and of the stress force across the sphere surface, i.e.,

$$\mathbf{u}^o = \mathbf{u}^i, \quad \mathbf{n}\cdot\tau^o = \mathbf{n}\cdot\tau^i \quad at \quad r=1 \tag{3b}$$

where $\mathbf{n}$ is the unit normal to the sphere surface.

We now derive a general solution of the Stokes equation (1a) and Brinkman equation (2a) plus the continuity equations (1b) and (2b) in terms of the fundamental eigenfunctions for a spherical coordinate system(r, ϕ, θ). According to (1) and (2) the pressure fields, $p^o(\mathbf{x})$ and $p^i(\mathbf{x})$ are harmonic functions, i.e.,

$$\nabla^2 p^o = \nabla^2 p^i = 0 \tag{4}$$

and can therefore be expressed in terms of solid spherical harmonics. We shall now specialize the general solution to separate domains involving the regions interior and exterior to the porous sphere.

General solution exterior to the sphere

For the situation in which the disturbance flow due to the presence of the particle is required to vanish at infinity [i.e., boundary condition (3b)], Lamb [16] outlines a general solution of the creeping motion equations in a series of solid spherical harmonics:

$$p^o(\mathbf{x}) = P^\infty(\mathbf{x}) + \sum_{n=1}^{\infty} \frac{p_n^o}{r^{2n+1}} \tag{5a}$$

and

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \sum_{n=1}^{\infty} \Big[\frac{\nabla \chi_n^o \times \mathbf{x}}{r^{2n+1}} + \nabla \frac{\phi_n^o}{r^{2n+1}} - \frac{n-2}{2n(2n-1)} r^2 \nabla \frac{p_n^o}{r^{2n+1}} + \frac{n+1}{n(2n-1)} \mathbf{x} \frac{p_n^o}{r^{2n+1}} \Big] \tag{5b}$$

where p_n^o, χ_n^o and ϕ_n^o are solid spherical harmonics of positive order n.

We now evaluate the stress vector (i.e., surface force, $\mathbf{n}\cdot\tau$) acting on the surface of a sphere in order to determine the flow fields from satisfying the boundary condition (3b). The stress vector $\mathbf{n}\cdot\tau$ on the sphere, in general, can be expressed as

$$\mathbf{n}\cdot\tau = -\frac{\mathbf{x}}{r} p + \Big(\frac{\partial \mathbf{u}}{\partial r} - \frac{\mathbf{u}}{r}\Big) + \frac{1}{r} \nabla (\mathbf{x}\cdot\mathbf{u}) \tag{6}$$

for an incompressible Newtonian fluid [cf. Happel and Brenner [17]]. By means of the general solution (5), Equation (6) can ultimately be expressed in the form:

$$\mathbf{n}\cdot\tau^o(\mathbf{x}) = \mathbf{n}\cdot\tau^\infty(\mathbf{x}) + \frac{1}{r}\sum_{n=1}^{\infty} \Big[-(n+2) \frac{\nabla \chi_n^o \times \mathbf{x}}{r^{2n+1}} - 2(n+2) \nabla \frac{\phi_n^o}{r^{2n+1}} - \frac{2n^2+1}{n(2n-1)} \mathbf{x} \frac{p_n^o}{r^{2n+1}} + \frac{n^2-1}{n(2n-1)} r^2 \nabla \frac{p_n^o}{r^{2n+1}} \Big] \tag{7}$$

where $\tau^\infty(\mathbf{x})$ is the stress field generated by $\mathbf{U}^\infty(\mathbf{x})$ and $P^\infty(\mathbf{x})$.

General solution interior to the sphere

A general solution of the Brinkman equation plus the continuity equation can be also derived in terms of solid spherical harmonics p_n^i, χ_n^i and ϕ_n^i of positive order n and the modified spherical Bessel function $\sqrt{\frac{\pi}{2\sigma r}} I_{n+\frac{1}{2}}(\sigma r)$ of order $n+\frac{1}{2}$ (the condition of boundedness of the velocity at origin limits us to positive harmonics):

$$p^i(\mathbf{x}) = \sum_{n=1}^{\infty} p_n^i \tag{8a}$$

and

$$\mathbf{u}^i(\mathbf{x}) = \sum_{n=1}^{\infty} \Big[-\frac{1}{\sigma^2} \nabla p_n^i - \psi_n(\sigma r) \nabla \chi_n^i \times \mathbf{x} + \{(n+1)\psi_{n-1}(\sigma r) + n\psi_{n+1}(\sigma r)\sigma^2 r^2\} \nabla \phi_n^i - n(2n+1)\psi_{n+1}(\sigma r)\sigma^2 \mathbf{x} \phi_n^i \Big] \tag{8b}$$

in which σ is defined by $\frac{1}{\sigma^2} = \frac{\kappa}{a^2}$, the dimensionless permeability. Here,

$$\psi_n(\sigma r) = \sqrt{\frac{\pi}{2}} (\sigma r)^{-(n+\frac{1}{2})} I_{n+\frac{1}{2}}(\sigma r) \tag{9a}$$

has the special properties:

$$\psi_0(\zeta) = \frac{\sinh \zeta}{\zeta}, \quad \psi_n(\zeta) = \frac{1}{\zeta} \psi'_{n-1}(\zeta) \tag{9b}$$

The stress vector $\mathbf{n}\cdot\tau^i$ interior to the sphere can also be expressed by utilizing the general solution (8) in combination with (6):

$$\mathbf{n}\cdot\tau^i(\mathbf{x}) = \frac{1}{r}\sum_{n=1}^{\infty} \Big[-Q_n(\sigma r) \nabla \chi_n^i \times \mathbf{x} - \frac{2}{\sigma^2}(n-1)\nabla p_n^i - p_n^i \mathbf{x} + R_n(\sigma r)\nabla \phi_n^i - \frac{2n+1}{r^2} S_n(\sigma r) \phi_n^i \mathbf{x} \Big] \tag{10a}$$

where

$$Q_n(\zeta) = \zeta^2 \psi_{n+1}(\zeta) + (n-1)\psi_n(\zeta) \tag{10b}$$

$$R_n(\zeta) = (n+1)\{\zeta^2 \psi_n(\zeta) + 2(n-1)\psi_{n-1}(\zeta)\} + n\zeta^2\{\zeta^2 \psi_{n+2}(\zeta) - \psi_{n+1}(\zeta)\} \tag{10c}$$

and

$$S_n(\zeta) = n\zeta^2\{\zeta^2 \psi_{n+2}(\zeta) - \psi_{n+1}(\zeta)\}. \tag{10d}$$

It should be noted that the general solutions defined by (5) and (8) automatically satisfy the governing differential equations, as well as the condition (3a) of vanishing disturbances in the far field. All that remains is to satisfy the boundary condition (3b) at the sphere surface, according to which the fluid velocity and surface force must be continuous across the surface.

This completes our derivation of the general solution forms for the flow fields both exterior and interior to the porous sphere. We shall turn shortly to the application of these solutions for the problem in which a porous sphere is immersed in a viscous fluid that is undergoing a mean flow at infinity. It is worthwhile studying the motion of porous particles in a mean flow, not only because it is interesting in its own right but also because the solution leads to a resolution of the general suspension rheology of entangled polymer molecules or flocs. The problem is also relevant to the resolution of the boundary effects on the particle motion via a normal reflections-type calculation procedure. When a particle moves in the vicinity of a fluid interface, the presence of the interface will induce a 'reflected' velocity field. The leading terms of the interface reflections include uniform streaming flows, linear shear and uniaxial straining flows, and quadratic paraboloidal [i.e., $\mathbf{U}^\infty(\mathbf{x}) = K(\xi y^2 + z^2)\mathbf{e}_x$] and stag-

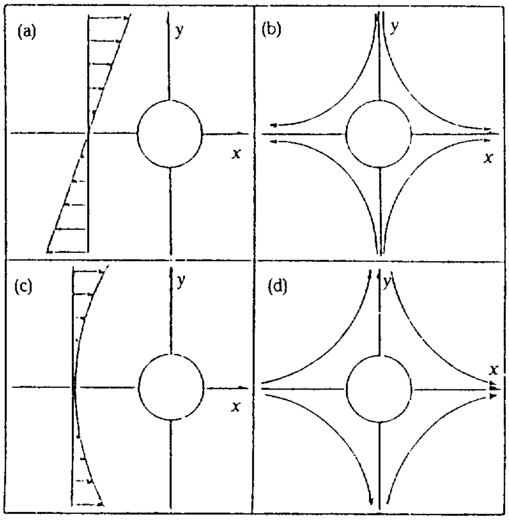

Fig. 1. Schematic diagrams for (a) a linear shear flow $\mathbf{U}^\infty(\mathbf{x}) = \Gamma\cdot\mathbf{x}$, (b) a uniaxial extensional flow $\mathbf{U}^\infty(\mathbf{x}) = \mathbf{E}\cdot\mathbf{x}$, (c) a quadratic paraboloidal flow $\mathbf{U}^\infty(\mathbf{x}) = K(\xi y^2 + z^2)\mathbf{e}_x$ and (d) a quadratic stagnation-like flow $\mathbf{U}^\infty(\mathbf{x}) = K[1/2(1+\xi)x^2\mathbf{e}_x - \xi xy\mathbf{e}_y - xy^2 e_z]$

nation-like [i.e., $\mathbf{U}^\infty(\mathbf{x}) = K\{\frac{1}{2}(1+\xi)x^2\mathbf{e}_x - \xi xy\mathbf{e}_y - xz\mathbf{e}_z\}$] flows [15]. Thus, in order to analyze the motion of particles near a plane interface using the reflections-method in conjunction with the fundamental singularity solutions for Stokes flow, we must determine the spatial distribution of fundamental singularities (e.g., Stokeslets, rotlets, stresslets and higher order singularities) that generates the same disturbance motion as the presence of the particle in an *unbounded* fluid domain [14, 15]. In the following, we shall examine the case of porous sphere which moves either in a linear flow or in a quadratic flow as depicted in Figure 1. The solution will then be expressed in terms of the fundamental singularity solutions of creeping motion equation, which will be used in the forthcoming part 2 of the present paper to study particle motions in the presence of a flat fluid interface. First, however, in view of the linearity of the problem it is necessary to determine the general motion of a porous particle in a quiescent fluid in order to analyze an arbitrary motion in a mean flow at infinity.

TRANSLATION AND ROTATION IN A QUIESCENT FLUID

Let us now consider the specific problem of a porous sphere which is translating or rotating in a quiescent fluid. We choose, for convenience, a moving coordinate system in which the particle is at rest with center of mass at origin. In it, a uniform streaming flow, $\mathbf{U}^\infty(\mathbf{x}) = -\mathbf{e}_x$, or a rotational flow, $\mathbf{U}^\infty(\mathbf{x}) = -\mathbf{e}_x \times \mathbf{x}$, around a stationary particle is precisely equivalent to translation ($\mathbf{U} = \mathbf{e}_x$) or rotation ($\Omega = \mathbf{e}_x$) with respect to the fixed frame of reference in a quiescent fluid. It should be noted that the undisturbed flow velocity $\mathbf{U}^\infty(\mathbf{x})$ is normalized by the translational or angular velocity, $\mathbf{U}$ or Ω, of the particle in the fluid at rest at infinity. Owing to the linearity of the problem, fisrt, we can analyze the streaming flow past a porous sphere, separately from the rotational flow, by utilizing the general solution forms (5), (7), (8) and (10). The most convenient method for doing this is to make the following substitutions into the general solution forms:

$$\{p_n, \phi_n, \chi_n\} = r^n \sum_{m=-n}^{n} \{A_{n,m}, B_{n,m}, C_{n,m}\} Y_{n,m}(\theta, \phi) \tag{11}$$

where $Y_{n,m}(\theta,\phi)$ is the normalized surface spherical harmonic of order n and rank m. All that is required is a specification of the undisturbed velocity $\mathbf{U}^\infty(\mathbf{x})$ and pressure $P^\infty(\mathbf{x})$ in terms of spherical harmonics; namely, the coefficients $A_{n,m}$, $B_{n,m}$ and $C_{n,m}$ in (11), and solution of the resulting algebraic relationships from the boundary condition (3b) at the particle surface. The *nonzero* spherical harmonics, determined from the boundary condition (3b), are

$$\begin{Bmatrix} p_1^o \\ p_1^i \end{Bmatrix} = \begin{Bmatrix} A_{1,0}^o(\sigma) \\ A_{1,0}^i(\sigma) \end{Bmatrix} r\ Y_{1,0}(\theta; \phi) \tag{12a}$$

$$\begin{Bmatrix} \phi_1^o \\ \phi_1^i \end{Bmatrix} = \begin{Bmatrix} B_{1,0}^o(\sigma) \\ B_{1,0}^i(\sigma) \end{Bmatrix} r\ Y_{1,0}(\theta, \phi) \tag{12b}$$

where

$$A_{1,0}^o(\sigma) = A_{1,0}^i(\sigma) = \frac{3\sigma^2\psi_1(\sigma)}{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)} \tag{12c, d}$$

$$B_{1,0}^o(\sigma) = -\frac{\sigma^2\{2\psi_2(\sigma) - \psi_1(\sigma)\}}{2\{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)\}} \tag{12e}$$

and

$$B_{1,0}^i(\sigma) = -\frac{1}{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)}. \tag{12f}$$

The solution, (12a,b), specifies completely the velocity and pressure fields exterior and interior to the porous particle. In particular, the velocity field exterior to the sphere is thus given by

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \frac{1}{2} A_{1,0}^o(\sigma)\left(\frac{\mathbf{e}_x}{r} + \frac{x\mathbf{x}}{r^3}\right)$$

$$+B^o_{1,0}(\sigma)\left(\frac{\mathbf{e}_x}{r^3}-\frac{3x\mathbf{x}}{r^5}\right). \tag{13}$$

It can be easily seen that the disturbance velocity due to the presence of the particle in (13) is precisely the same as that induced by the Stokeslet $\alpha=\frac{1}{2}A^o_{1,0}(\sigma)\mathbf{e}_x$ and the potential dipole $\beta=-B^o_{1,0}(\sigma)\mathbf{e}_x$ located at the sphere center. As $\sigma\to\infty$, Equation (13) reduces to the velocity field for the case of a rigid impermeable sphere, and is identical with the flow generated by the singularities, $\alpha=\frac{3}{4}\mathbf{e}_x$ and $\beta=-\frac{1}{4}\mathbf{e}_x$ at origin [cf. Chwang and Wu [18]].

As there is no contribution to the drag force from potential doublet, the drag on the sphere is simply given as:

$$\mathbf{F}=-4\pi A^o_{1,0}(\sigma)\mathbf{e}_x \tag{14}$$

(The dimensional drag force is **F** multiplied by μUa). When either $\sigma\to 0$ or $\sigma\to\infty$, i.e., in the limit of large or small permeability, the leading terms in (14) can be determined from the asymptotic properties of modified spherical Bessel functions:

$$\mathbf{F}=-\frac{4}{3}\pi\sigma^2\{1-\frac{4}{15}\sigma^2+O(\sigma^4)\}\mathbf{e}_x \quad as \quad \sigma\to 0 \tag{15}$$

and

$$\mathbf{F}=-6\pi\{1-\frac{1}{\sigma}+O(\sigma^{-2})\}\mathbf{e}_x \quad as \quad \sigma\to\infty \tag{16}$$

which are identical with the asymptotic solutions of Kojima [13]. In Figure 2, the drag of (14) is plotted as a function of the dimensionless permeability. The drag asymptotically calculated by Kojima is also shown in the figure. It can be seen that the approximation solution shows reasonably good agreement with the exact solution over the entire range of permeability.

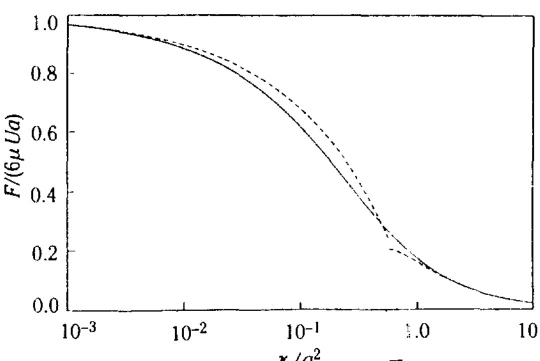

Fig. 2. Dimensionless drag force, $\frac{F}{6\pi\mu Ua}$, as a function of the dimensionless permeability $\frac{\kappa}{a^2}$; ——, for the exact solution of (14); - - - -, for the asymptotic solutions of (15,16).

We now turn to the case of a stationary sphere immersed in the rotating fluid with a constant angular velocity $\Omega=-\mathbf{e}_x$ (i.e., $\mathbf{U}^\infty(\mathbf{x})=-\mathbf{e}_x\times\mathbf{x}$). As in the preceding example, the solution that satisfies the boundary condition at the sphere surface can be simply represented by spherical harmonics

$$\begin{Bmatrix}\chi^o_1\\ \chi^i_1\end{Bmatrix}=\begin{Bmatrix}C^o_{1,0}(\sigma)\\ C^i_{1,0}(\sigma)\end{Bmatrix} r\; Y_{1,0}(\theta,\phi) \tag{17a}$$

where

$$C^o_{1,0}(\sigma)=\frac{\sigma^2\psi_2(\sigma)}{\psi_0(\sigma)} \text{ and } C^i_{1,0}(\sigma)=\frac{3}{\psi_0(\sigma)} \tag{17b, c}$$

All other spherical harmonics in the general solution forms, (5), (7), (8) and (10), are zero. The velocity field exterior to the sphere, given by the spherical harmonics χ^o_1, is

$$\mathbf{u}^o(\mathbf{x})=-(\mathbf{e}_x\times\mathbf{x})\left\{1-\frac{C^o_{1,0}(\sigma)}{r^3}\right\} \tag{18}$$

in which the disturbance velocity can be also generated by a rotlet $\gamma=C^o_{1,0}(\sigma)\mathbf{e}_x$, at the sphere center. It should be noted that, when $\sigma\to\infty$, the solution (18) reduces to the case of a rigid impermeable sphere (i.e., $C^o_{1,0}(\sigma)\to 1$).

The torque exerted on the sphere by the rotational flow can be easily calculated from the strength of the rotlet:

$$\mathbf{T}=-8\pi C^o_{1,0}(\sigma)\mathbf{e}_x \tag{19}$$

The dimensional torque is given by **T** multiplied by the factor $\mu\Omega a^3$. In the limit of small permeability (i.e., σ ∞) one obtains for the asmptotic behavior

$$\mathbf{T}=-8\pi\left\{1-\frac{3}{\sigma}+O(\sigma^{-2})\right\}\mathbf{e}_x \quad as \quad \sigma\to\infty \tag{20a}$$

and for large permeabilities, σ—0, one has

$$\mathbf{T}=-\frac{8}{15}\pi\sigma^2\left\{1-\frac{2}{21}\sigma^2+O(\sigma^4)\right\}\mathbf{e}_x \quad as \quad \sigma\to 0. \tag{20b}$$

The present solution (19) agrees with that in Felderhof and Deutch [2] based on an *ad hoc* ansatz involving separation of variables in axisymmetric spherical coordinate system (i.e., $\frac{\partial}{\partial\phi}=0$). In Figure 3, the hydrodynamic torque, (19), is plotted as a function of the dimensionless permeability. Also shown for comparison is the corresponding approximation solution of (20a,b). There is very good agreement between the two solutions in the limit of large of small permeability.

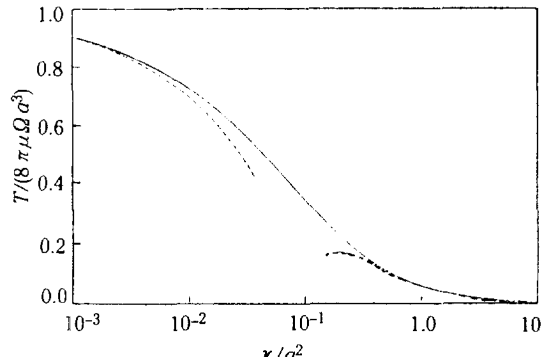

Fig. 3. Dimensionless hydrodynamic torque $\frac{T}{8\pi\mu\Omega a^3}$ as a function of the dimensionless permeability $\frac{\chi}{a^2}$; ——, for the exact solution of (19); · · · ·, for the asymptotic solutions of (20a,b).

LINEAR FLOW

Axisymmetric straining flow

We consider the simplest case of an extensional flow (also called 'hyperbolic flow') past the present porous sphere with an axisymmetric free stream

$$\mathbf{U}^\infty(\mathbf{x}) = \mathbf{L}\cdot\mathbf{x}$$

where the strain rate tensor $\mathbf{L} = \{L_{ij}\}$ is defined by $L_{ij} = 3\delta_{i1}\delta_{j1} - \delta_{ij}$ and nondimensionalized with respect to the strain rate L (i.e., $u_c = La$). In this case, expanding $\mathbf{U}^\infty(\mathbf{x})$ and $P^\infty(\mathbf{x})$ in terms of spherical harmonics, it can be easily shown that all terms in the general solutions, (5), (7), (8) and (10), vanish except for $n = 2$, $m = 0$ and $\chi_2 = 0$. The coefficients $A_{2,0}$, $B_{2,0}$ can be calculated from satisfying the boundary condition at the sphere surface:

$$\begin{Bmatrix} p_2^o \\ p_2^i \end{Bmatrix} = \begin{Bmatrix} A_{2,0}^o(\sigma) \\ A_{2,0}^i(\sigma) \end{Bmatrix} r^2\, Y_{2,0}(\theta,\phi) \tag{21a}$$

$$\begin{Bmatrix} \phi_2^o \\ \phi_2^i \end{Bmatrix} = \begin{Bmatrix} B_{2,0}^o(\sigma) \\ B_{2,0}^i(\sigma) \end{Bmatrix} r^2\, Y_{2,0}(\theta,\phi) \tag{21b}$$

where

$$A_{2,0}^o(\sigma) = A_{2,0}^i(\sigma) = -\frac{10\sigma^2\,\psi_2(\sigma)}{\psi_0(\sigma)+10\,\psi_2(\sigma)} \tag{21c, d}$$

$$B_{2,0}^o(\sigma) = \frac{1}{3}\cdot\frac{2\,\psi_0(\sigma) - 30\,\psi_2(\sigma) - 5\sigma^2\,\psi_2(\sigma)}{\psi_0(\sigma)+10\,\psi_2(\sigma)} \tag{21e}$$

and

$$B_{2,0}^i(\sigma) = \frac{1}{\psi_0(\sigma)+10\,\psi_2(\sigma)}\,. \tag{21f}$$

The fundamental singularities required in constructing the present exact solution for the velocity field *exterior* to the porous sphere,

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \frac{1}{2}A_{2,0}^o(\sigma)\,r^{-3}\,Y_{2,0}(\theta,\phi)\,\mathbf{x} + B_{2,0}^o(\sigma)\nabla\{r^{-3}\,Y_{2,0}(\theta,\phi)\}, \tag{22}$$

are easily seen to be a stresslet and potential quadrupole of the form:

Stresslet : $\frac{1}{4}A_{2,0}^o(\sigma)\,\mathbf{u}_{SS}(\mathbf{x}\,;\,\mathbf{e}_x,\mathbf{e}_y)$

Potential Quadrupole ; $\frac{1}{2}B_{2,0}^o\,\mathbf{u}_{PQ}(\mathbf{x}\,;\,\mathbf{e}_x,\mathbf{e}_y)$.

[cf. Chwang and Wu [18] for the fundamental solution forms of $\mathbf{u}_{SS}$ and $\mathbf{u}_{PQ}$]. It should be noted that, when $\sigma\to\infty$, the present solution for the velocity field exterior to the sphere reduces to

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) - \frac{5}{2}\mathbf{u}_{SS}(\mathbf{x}\,;\,\mathbf{e}_x,\mathbf{e}_x) - \frac{1}{2}\mathbf{u}_{PQ}(\mathbf{x}\,;\,\mathbf{e}_x,\mathbf{e}_x) \tag{23}$$

which is identical with Chwang and Wu's [18] result for the case of a rigid impermeable sphere.

All of the preceding examples are concerned with a porous sphere immersed in an *axisymmetric* undisturbed flow, and thus the rank m of the spherical harmonics is consequently zero. We now proceed to consider non-axisymmetric undisturbed flows (e.g., simple shear flow) with nontrivial azimuthal dependence (i.e., $\frac{\partial}{\partial\phi} \neq 0$).

Liner Shear flow

An incompressible Newtonian fluid is in steady shear flow past a neutrally buoyant porous sphere which is freely suspended in the fluid. The fluid velocity at infinity, nondimensionalzed with respect to $u_c = \Gamma a$ (Γ: shear rate),

$$\mathbf{U}^\infty(\mathbf{x}) = y\mathbf{e}_x\,. \tag{24}$$

The case in which $\mathbf{U}^\infty(\mathbf{x}) \neq \mathbf{O}$ at the sphere center can be treated by superimposing a uniform streaming flow past a sphere, $\mathbf{U}^\infty(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{O})$ on the simple shear flow.

In the general solutions of Stokes equation, (5) and (7), and of Brinkman equation, (8) and (10), the nonzero terms in this case are χ_1 (with rank $m = -1$), p_2 and ϕ_2 (each with rank $m = 1$). The coefficients $A_{2,1}$, $B_{2,1}$ and $C_{1,-1}$ are obtained by solving the resulting algebraic equations from the boundary condition (3b):

$$\begin{Bmatrix} p_2^o \\ p_2^i \end{Bmatrix} = \begin{Bmatrix} A_{2,1}^o(\sigma) \\ A_{2,1}^i(\sigma) \end{Bmatrix} r^2\, Y_{2,1}(\theta,\phi) \tag{25a}$$

$$\begin{Bmatrix} \phi_2^o \\ \phi_2^i \end{Bmatrix} = \begin{Bmatrix} B_{2,1}^o(\sigma) \\ B_{2,1}^i(\sigma) \end{Bmatrix} r^2 Y_{2,1}(\theta, \phi) \qquad (25b)$$

and

$$\begin{Bmatrix} \chi_1^o \\ \chi_1^i \end{Bmatrix} = \begin{Bmatrix} C_{1,-1}^o(\sigma) \\ C_{1,-1}^i(\sigma) \end{Bmatrix} r\, Y_{1,-1}(\theta, \phi) \qquad (25c)$$

where

$$A_{2,1}^o(\sigma) = A_{2,1}^i(\sigma) = -\frac{5\sigma^2 \psi_2(\sigma)}{3\{\psi_0(\sigma) + 10\,\psi_2(\sigma)\}} \qquad (25d, e)$$

$$B_{2,1}^o(\sigma) = \frac{\sigma^2\{2\,\psi_3(\sigma) - \psi_2(\sigma)\}}{6\{\psi_0(\sigma) + 10\,\psi_2(\sigma)\}} \qquad (25f)$$

$$B_{2,1}^i(\sigma) = \frac{1}{6\{\psi_0(\sigma) + 10\,\psi_2(\sigma)\}} \qquad (25g)$$

$$C_{1,-1}^o(\sigma) = \frac{\sigma^2\,\psi_2(\sigma)}{2\,\psi_0(\sigma)} \qquad (25h)$$

and

$$C_{1,-1}^i(\sigma) = \frac{3}{2\,\psi_0(\sigma)}. \qquad (25i)$$

The velocity field exterior to the sphere can be readily evaluated by substituting (25) into the general solution (5) with all other spherical harmonics taken to be zero:

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + C_{1,-1}^o(\sigma)\frac{\mathbf{e}_z \times \mathbf{x}}{r^3} + \frac{3}{2}A_{2,1}^o(\sigma)\frac{xy}{r^5}\mathbf{x} + 3B_{2,1}^o(\sigma)\nabla\frac{xy}{r^5}. \qquad (26)$$

It should be easily verified that the types of singularities required for the construction of the solution, (26), apart from the primary flow are a rotlet, a stresslet and a potential quadrupole of the form:

Rotlet : $C_{1,-1}^o(\sigma)\,\mathbf{u}_R(\mathbf{x}\,;\mathbf{e}_z)$

Stresslet : $\frac{1}{2}A_{2,1}^o(\sigma)\,\mathbf{u}_{SS}(\mathbf{x}\,;\mathbf{e}_x, \mathbf{e}_y)$

Potential Quadrupole : $B_{2,1}^o(\sigma)\,\mathbf{u}_{SS}(\mathbf{x}\,;\mathbf{e}_x, \mathbf{e}_y)$.

The dimensionless hydrodynamic torque on a porous sphere in the uniform shear flow can be evaluated from the rotlet and is equal to

$$\mathbf{T} = -4\pi C_{1,-1}^o(\sigma)\,\mathbf{e}_z. \qquad (27)$$

The dimensional torque is $\mathbf{T}$ multiplied by $\mu\Gamma a^3$. This is the magnitude of the torque that is required to keep the sphere from rotating. Thus, a freely suspended particle will rotate with an angular velocity that can be easily determined from (19) and (27):

$$\Omega = \frac{1}{2}\mathbf{e}_z \qquad (28)$$

being nondimensionalized by $u_c = \Gamma a$. It is noteworthy that the angular velocity is $\frac{1}{2}$ of the vorticity vector in the primary flow irrespective of the permeability of the particle.

In the limiting case of an impermeable sphere (i.e. $\sigma \to \infty$), the solution (26) reduces to

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \frac{1}{2}\mathbf{u}_R(\mathbf{x}\,;\mathbf{e}_z) - \frac{5}{6}\mathbf{u}_{SS}(\mathbf{x}\,;\mathbf{e}_x, \mathbf{e}_y) - \frac{1}{6}\mathbf{u}_{PQ}(\mathbf{x}\,;\mathbf{e}_x, \mathbf{e}_y)$$

and by (27),

$$\mathbf{T} = -4\pi\mathbf{e}_z$$

all being well-known results [cf. Burgers [19] and Chwang and Wu [18]].

QUADRATIC FLOW

Paraboloidal flow

As a further variation of the free stream, we consider a flow with a paraboloidal velocity profile

$$\mathbf{U}^\infty(\mathbf{x}) = (\xi y^2 + z^2)\mathbf{e}_x,\ \mathbf{P}^\infty(\mathbf{x}) = 2(\xi + 1)x \quad (29a, b)$$

past a porous sphere of radius a, centered at origin (in this case $u_c = Ka^2$, $p_c = \mu Ka$, with proportionality constant K). The paraboloidal flow may be either elliptic or hyperbolic depending on the sign of the parameter ξ. When $\xi = 0$, the paraboloidal flow degenerates into a 2-dimensional flow. For $\xi > 0$, it represents Hagen-Poiseuille flow through a tube of elliptic cross-section. Hyperbolic paraboloidal flow ($\xi < 0$) may not exist physically, but it can certainly serve as a local component of complicated flow. An off-centered paraboloidal profile is equivalent to a centered one superimposed on a uniform streaming flow plus a linear shear flow.

Let us, first, consider a simple case of an axisymmetric paraboloidal flow with $\xi = 1$. In view of the axisymmetric nature of the problem, it is clear that the solution must be independent of the azimuthal angle ϕ, so that the only nonzero coefficients in the general solution are those with $m = 0$. In addition, $\chi_n = 0$ for all m. The remaining spherical harmonics can be determined from the boundary condition (3b) at the sphere surface in combination with the prescribed flow field at infinity that is incorporated into the general solutions, (5) and (8). One eventually obtains

$$\begin{Bmatrix} p_n^o \\ p_n^i \end{Bmatrix} = \begin{Bmatrix} D_{n,0}^o(\sigma) \\ D_{n,0}^i(\sigma) \end{Bmatrix} r^n Y_{n,0}(\theta, \phi),\ n = 1 \text{ and } 3 \qquad (30a)$$

$$\begin{Bmatrix} \phi_n^o \\ \phi_n^i \end{Bmatrix} = \begin{Bmatrix} E_{n,0}^o(\sigma) \\ E_{n,0}^i(\sigma) \end{Bmatrix} r^n Y_{n,0}(\theta, \phi),\ n = 1 \text{ and } 3 \qquad (30b)$$

where

$$D^o_{1,0}(\sigma) = \frac{2\sigma^2\{2\psi_2(\sigma) - \psi_1(\sigma)\}}{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)}$$

$$= D^i_{1,0}(\sigma) - 4 \quad (30c, d)$$

$$D^o_{3,0}(\sigma) = \frac{2\sigma^2\psi_3(\sigma)}{15\psi_3(\sigma) + \frac{4}{7}\psi_1(\sigma)} = D^i_{3,0}(\sigma) \quad (30e, f)$$

$$E^o_{1,0}(\sigma) = \frac{3\{4\psi_0(\sigma) - 14\psi_1(\sigma)}{5\{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)\}} - \frac{\sigma^2\psi_1(\sigma) + 10\psi_2(\sigma)\}}{5\{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)\}} \quad (30g)$$

$$E^i_{1,0}(\sigma) = \frac{2(3+\sigma^2)}{\sigma^2\{2\psi_0(\sigma) + 2\sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)\}} \quad (30h)$$

$$E^o_{3,0}(\sigma) = \frac{\sigma^2\{\psi_3(\sigma) - 2\psi_4(\sigma)\}}{7\{15\psi_3(\sigma) + \frac{4}{7}\psi_1(\sigma)\}} \quad (30i)$$

and

$$E^i_{3,0}(\sigma) = -\frac{2}{21\{15\psi_3(\sigma) + \frac{4}{7}\psi_1(\sigma)\}} \quad (30j)$$

The velocity field exterior to the sphere, corresponding to the exact solution (30), can be expressed in terms of the fundamental singularity solutions of Stokes flow. It is a simple matter to determine the required singularities that generate the disturbance flow due to the presence of a porous sphere immersed in the primary flow $\mathbf{U}^\infty(\mathbf{x}) = (y^2 + z^2)\mathbf{e}_x$:

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \frac{1}{2}\{D^o_{1,0}(\sigma) + \frac{1}{6}D^o_{3,0}(\sigma)\frac{\partial^2}{\partial x^2}\}\mathbf{u}_S(\mathbf{x}; \mathbf{e}_x) + \{\frac{1}{30}D^o_{3,0}(\sigma) - E^o_{1,0}(\sigma) - \frac{1}{6}E^o_{3,0}(\sigma)\frac{\partial^2}{\partial x^2}\}\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) \quad (31)$$

in which $\mathbf{u}_s(\mathbf{x}; \mathbf{e}_x)$ and $\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x)$ denote the fundamental solutions for a Stokeslet and a potential doublet located at origin. As suggested by the variable velocity gradient of the primary flow we need an axial Stokes quadrupole $\frac{\partial^2}{\partial x^2}\mathbf{u}_s(\mathbf{x}; \mathbf{e}_x)$ and a potential octupole $\frac{\partial^2}{\partial x^2}\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x)$ that is associated with the Stokes quadrupole to balance the power-law variations of the solution in r [cf. Chwang [20]].

Although the primary flow has a pressure gradient, $\nabla P^\infty(\mathbf{x}) = 4\mathbf{e}_x$ hence producing a 'buoyancy effect' on the sphere, this buoyancy force must be balanced by the net effect of the viscous stress, $\nabla^2\mathbf{U}^\infty(\mathbf{x}) = 4\mathbf{e}_x$, of the primary flow. The drag on the sphere therefore comes solely from the contribution of the Stokeslet:

$$\mathbf{F} = -4\pi D^o_{1,0}(\sigma)\mathbf{e}_x \quad (32)$$

(the dimensional drag is $\mathbf{F}$ multiplied by μKa^3). For a rigid impermeable sphere, Chwang and Wu [18] discovered the types of singularities necessary to construct the solution and evaluated the drag force from the Stokeslet distribution, all being identical with the present solutions (31) and (32), in the limit of $\sigma \to \infty$. It is noteworthy that the primary flow with Hagen-Poiseuille velocity profile in a circular tube of radius R can be treated by superimposing a uniform streaming flow $\mathbf{U}^\infty(\mathbf{x}) = R^2\mathbf{e}_x$ on a paraboloidal flow $\mathbf{U}^\infty(\mathbf{x}) = -(y^2 + z^2)\mathbf{e}_x$.

Let us now consider a more general paraboloidal flow, (29) with $\xi \neq 0$, past a porous sphere. It is simple matter to construct an exact solution for this problem. All we have to do for this is to determine a solution either for

$$\mathbf{U}^\infty(\mathbf{x}) = y^2\mathbf{e}_x \quad (33a)$$

or for

$$\mathbf{U}^\infty(\mathbf{x}) = z^2\mathbf{e}_x \quad (33b)$$

owing to the linearity of the problem (i.e., superposition rule). However, the solution for (33a) [or (33b)] can be easily obtained by decomposing the exact solution (31) for $\mathbf{U}^\infty(\mathbf{x}) = (y^2 + z^2)\mathbf{e}_x$ into the two *symmetric* parts for 2-dimensional paraboloidal flows of (33a,b), in particular, for the primary flow $\mathbf{U}^\infty(\mathbf{x}) = y^2\mathbf{e}_x$, the velocity field exterior to the sphere is given by

$$\mathbf{u}^o(\mathbf{x}) = y^2\mathbf{e}_x + \frac{1}{4}D^o_{1,0}(\sigma)\mathbf{u}_S(\mathbf{x}; \mathbf{e}_x) + \frac{1}{12}D^o_{3,0}(\sigma)\frac{\partial}{\partial y}\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_y) + \{E^o_{1,0}(\sigma) + \frac{1}{20}D^o_{3,0}(\sigma)\}\frac{\partial}{\partial y}\mathbf{u}_R(\mathbf{x}; \mathbf{e}_z) + \frac{1}{6}E^o_{3,0}(\sigma)\frac{\partial^2}{\partial y^2}\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) \quad (34)$$

In obtaining (34) from (31), we have applied the identities:

$$\frac{\partial}{\partial y}\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_y) + \frac{\partial}{\partial z}\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_z) = \frac{\partial^2}{\partial x^2}\mathbf{u}_S(\mathbf{x}; \mathbf{e}_x) + \mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) \quad (35a)$$

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) = -\frac{\partial^2}{\partial x^2}\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) \quad (35b)$$

and

$$\frac{\partial}{\partial z}\mathbf{u}_R(\mathbf{x}; \mathbf{e}_y) - \frac{\partial}{\partial y}\mathbf{u}_R(\mathbf{x}, \mathbf{e}_z) = \mathbf{u}_D(\mathbf{x}; \mathbf{e}_x) \quad (35c)$$

As a matter of fact, the solution for $\mathbf{U}^\infty(\mathbf{x}) = z^2\mathbf{e}_x$ is the counterpart of the expression (34), with $\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}, \mathbf{e}_z$ and $\mathbf{e}_y$ replaced by $\frac{\partial}{\partial z}, \frac{\partial^2}{\partial z^2}, -\mathbf{e}_y$ and $\mathbf{e}_z$, respectively. The complete solution for (29) with an arbitrary ξ may be constructed by applying the superposition rule to the solution for each two-dimensional paraboloidal flow (33a,b). For instance, the total hydrodynamic force acting on the porous sphere in the primary flow, (29), can be obtained as:

$$\mathbf{F} = -2\pi(1+\xi)D^o_{1,0}(\sigma)\mathbf{e}_x. \qquad (36)$$

In this case the torque $\mathbf{T}$ is obviously zero. The drag, (36), may be regarded as associated with that on the same sphere in a uniform streaming flow of an equivalent velocity

$$\mathbf{U}_e^\infty = \frac{(1+\xi)\{\psi_1(\sigma) - 2\psi_2(\sigma)\}}{3\,\psi_1(\sigma)}\mathbf{e}_x \qquad (37)$$

[see (14) for the drag on a porous sphere in a uniform streaming flow]. Thus, a freely suspended porous sphere in the primary flow will translate with a velocity given by (37). It is of interest to note that the velocity (37) is different from the surface average of the primary flow velocity (29a). In Figure 4, the equivalent velocity $\mathbf{U}_e^\infty$ of (37) is plotted versus the dimensionless permeability. It can be noted that only for an impermeable sphere (i.e., $\sigma \to \infty$), does the equivalent velocity, $\mathbf{U}_e^\infty$ of (37), become the same as the surface average velocity, both having the value $\frac{1}{3}(1+\xi)\mathbf{e}_x$.

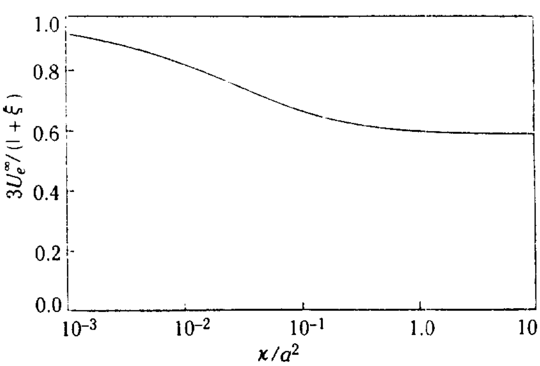

Fig. 4. Equivalent translational velocity, U_e^∞ of (37), as a function of the dimensionless permeability $\frac{\kappa}{a^2}$.

Stagnation-like quadratic flow

Let us now turn to a stagnation-like quadratic flow with a velocity profile

$$\mathbf{U}^\infty(\mathbf{x}) = \frac{1}{2}(1+\xi)x^2\,\mathbf{e}_x - \xi xy\mathbf{e}_y - xz\mathbf{e}_z \qquad (38a)$$

which obviously satisfies the creeping motion equation if the pressure associated with it is

$$P^\infty(\mathbf{x}) = (1+\xi)x. \qquad (38b)$$

The stagnation plane is defined by $x = 0$. Although this type of quadratic flow is of some intrinsic interest in serving as a local component of a more complicated flow in an *unbounded* domain, it plays an important role in determining a general motion of a particle near a flat interface.

We consider, first, a simple case of an axisymmetric stagnation-like flow, (38) with $\xi = 0$. In this case, the exact solution for the flow fields exterior and interior to the sphere involves the nonzero spherical harmonics, p_n and ϕ_n with $n = 1, 3$ and $m = 0$, in the general solutions:

$$\begin{Bmatrix} p_n^o \\ p_n^i \end{Bmatrix} = \frac{7-5n}{4}\begin{Bmatrix} D^o_{n,0}(\sigma) \\ D^i_{n,0}(\sigma) \end{Bmatrix} r^n\, Y_{n,0}(\theta,\phi), \quad n = 1 \text{ and } 3 \qquad (39a)$$

$$\begin{Bmatrix} \phi_n^o \\ \phi_n^i \end{Bmatrix} = \frac{7-5n}{4}\begin{Bmatrix} E^o_{n,0}(\sigma) \\ E^i_{n,0}(\sigma) \end{Bmatrix} r^n\, Y_{n,0}(\theta,\phi), \quad n = 1 \text{ and } 3 \qquad (39b)$$

Here, the coefficients, $D_{n,0}$ and $E_{n,0}$ are given by (30c-j) in the preceding problem for a paraboloidal flow.

The fundamental singularity solutions for Stokes flow can be used in an interesting way to construct the flow field exterior to the porous sphere that would be generated by the presence of the particle. The exterior velocity associated with the spherical harmonics p_n^o and ϕ_n^o of (39) can be represented by a Stokeslet (required to produce a drag), a potential dipole (associated with the Stokeslet to account for the body-thickness effect), an axial Stokes quadrupole (as suggested by the variable velocity gradient of $\mathbf{U}^\infty(\mathbf{x})$) and a potential octupole (associated with the Stokes quadrupole):

$$\mathbf{u}^o(\mathbf{x}) = \mathbf{U}^\infty(\mathbf{x}) + \left\{\frac{1}{4}D^o_{1,0}(\sigma) - \frac{1}{6}D^o_{3,0}(\sigma)\frac{\partial^2}{\partial x^2}\right\}\mathbf{u}_S(\mathbf{x};\mathbf{e}_x) - \left\{\frac{1}{15}D^o_{3,0}(\sigma) + \frac{1}{2}E^o_{1,0}(\sigma) - \frac{1}{3}E^o_{3,0}(\sigma)\frac{\partial^2}{\partial x^2}\right\}\mathbf{u}_D(\mathbf{x};\mathbf{e}_x). \qquad (40)$$

The total hydrodynamic force on the sphere is evaluated from the contribution of Stokeslet:

$$\mathbf{F} = -2\pi D^o_{1,0}(\sigma)\mathbf{e}_x \qquad (41)$$

which reduces in the limiting case of an impermeable sphere (i.e., $\sigma \to \infty$) to $\lim_{\sigma\to\infty}\mathbf{F} = 2\pi\mathbf{e}_x$.

Finally, we consider a more general quadratic flow, (38) with $\xi \neq 0$. The solution is an analogous to that of $\mathbf{U}^\infty(\mathbf{x}) = (\xi y^2 + z^2)\mathbf{e}_x$ in the previous example. In view of the linearity of the problem, it is sufficient to solve the primary flow

$$\mathbf{U}^\infty(\mathbf{x}) = \frac{1}{2}x^2\mathbf{e}_x - xy\mathbf{e}_y \qquad (42a)$$

or

$$U^{\infty}(\mathbf{x}) = \frac{1}{2} x^2 \mathbf{e}_x - xz\mathbf{e}_z \tag{42b}$$

in order to construct the exact solution for $\mathbf{U}^{\infty}(\mathbf{x})$ given by (38). However, if we note that the primary flow, (38) with $\xi = 1$, consists of two symmetric components of (42a,b), then decomposing the solution (40) into the two parts we can easily determine the velocity field for each component flow. The result for the 2-demensional stagnation-like flow, $\mathbf{U}^{\infty}(\mathbf{x}) = \frac{1}{2} x^2\mathbf{e}_x - xy\mathbf{e}_y$, is

$$\begin{aligned}\mathbf{u}^o(\mathbf{x}) = & \frac{1}{2} x^2 \mathbf{e}_x - xy\mathbf{e}_y + \{\frac{1}{8} D^o_{1,0}(\sigma) \\ & - \frac{1}{24} D^o_{3,0}(\sigma) \frac{\partial^2}{\partial x^2}\}\mathbf{u}_S(\mathbf{x}\ ;\ \mathbf{e}_x) + \frac{1}{2} E^o_{1,0}(\sigma) \\ & - \frac{1}{60} D^o_{3,0}(\sigma)\{\frac{\partial}{\partial x} \mathbf{u}_{SS}(\mathbf{x}\ ;\ \mathbf{e}_y, \mathbf{e}_y) \\ & - \{\frac{1}{15} D^o_{3,0}(\sigma) + \frac{1}{2} E^o_{1,0}(\sigma)\}\frac{\partial}{\partial y} \mathbf{u}_{SS}(\mathbf{x}\ ;\ \mathbf{e}_x, \mathbf{e}_y) \\ & + \frac{1}{12} E^o_{3,0}(\sigma)\{\frac{\partial^2}{\partial x^2} \mathbf{u}_D(\mathbf{x}\ ;\ \mathbf{e}_x) \\ & - 2\frac{\partial^2}{\partial x \partial y} \mathbf{u}_D(\mathbf{x}\ ;\ \mathbf{e}_y)\}\}.\end{aligned} \tag{43}$$

In obtaining the above expression for the velocity field around a porous sphere from (34), we have applied the identities, (30a,b) and

$$\frac{\partial}{\partial x}\mathbf{u}_{SS}(\mathbf{x}\ ;\ \mathbf{e}_y, \mathbf{e}_y) + \frac{\partial}{\partial x}\mathbf{u}_{SS}(\mathbf{x}\ ;\ \mathbf{e}_z, \mathbf{e}_z) = \frac{\partial^2}{\partial x^2}\mathbf{u}_S(\mathbf{x}\ ;\ \mathbf{e}_x) \tag{44a}$$

$$\frac{\partial^2}{\partial x \partial y}\mathbf{u}_D(\mathbf{x}\ ;\ \mathbf{e}_y) + \frac{\partial^2}{\partial x \partial z}\mathbf{u}_D(\mathbf{x}\ ;\ \mathbf{e}_z) = - \frac{\partial^2}{\partial x^2}\mathbf{u}_D(\mathbf{x}\ ;\ \mathbf{e}_x), \tag{44b}$$

between the fundamental singularity solution of creeping motion equation. Consequently, the solution for the 2-dimensional stagnation-like flow of (42b) can be immediately determined on replacing $\frac{\partial}{\partial y}$ and $\mathbf{e}_y$ in (43) by $\frac{\partial}{\partial z}$ and $\mathbf{e}_z$, respectively. This completes the solution for $\mathbf{U}^{\infty}(\mathbf{x})$ of (38) with an arbitrary ξ.

The hydrodynamic force acting on the sphere immersed in the primary flow, $\mathbf{U}^{\infty}(\mathbf{x})$ of (38) is thus given by

$$\mathbf{F} = - \pi(1-\xi) D^o_{1,0}(\sigma)\ \mathbf{e}_x . \tag{45}$$

When $\sigma \to \infty$, Equation (45) reduces to the drag for the case of an impermeable sphere, and is identical with the result of Chwang [20] for $\xi = 0$.

This completes the solution for a porous sphere immersed either in a linear shear and an axisymmetric straining flows or in a quadratic paraboloidal and a stagnation-like flows of (29) and (30). As we shall see in the forthcoming part 2 of the present paper, these solutions play an important role in determining the general motion of a porous particle in the vicinity of a plane fluid-fluid interface.

DISCUSSION

In the previous sections, we have analyzed the motion of a porous sphere through a linear or a quadratic mean flow at infinity in an unbounded single-fluid domain. The linearity of the problem enables us to determine the translational and angular velocities of a freely suspended neutrally buoyant porous sphere in the prescribed flow:

$$\mathbf{U} = \frac{\mathbf{F}}{4\pi A^o_{1,0}(\sigma)}, \quad \Omega = \frac{\mathbf{T}}{8\pi C^o_{1,0}(\sigma)} . \tag{46a, b}$$

Here, $\mathbf{F}$ and $\mathbf{T}$ are the hydrodynamic force and torque acting on a stationary porous sphere due to the existence of a linear or a quadratic flow at large distances from the particle. Thus, given the initial particle position, these equations (46a,b) provide its complete trajectory in the flow. As a simple illustration, we determine the trajectory for a porous sphere freely suspended in an off-centered paraboloidal flow, $\mathbf{U}^{\infty}(\mathbf{x}) = \{\xi(y-y_0)^2 + (z-z_0)^2\}\mathbf{e}_x$, that is equivalent to a centered one (i.e., with $y_0 = z_0 = 0$) superimposed on a uniform streaming flow plus linear shear flows. The result is

$$\mathbf{U} = \left(\xi y_0^2 + z_0^2 + \frac{1+\xi}{3} \cdot \frac{\{\psi_1(\sigma) - 2\psi_2(\sigma)\}}{\psi_1(\sigma)} \right)\mathbf{e}_x \tag{47a}$$

and

$$\Omega = \frac{1}{2}\ \xi y_0 \mathbf{e}_z - \frac{1}{2} z_0 \mathbf{e}_y . \tag{47b}$$

The trajectory equations (47a,b) are relevant to the problem of a porous sphere freely suspended at an arbitrary point in Poiseuillian flow through a cylindrical tube of elliptic cross-section.

It has been found that, in the case of dilute suspension, the contributions to the bulk stress from the various particles are independent, and the contributions arising from the bulk rate of strain can be characterized by a *stresslet* in the pure straining motion of the ambient fluid [cf. Batchelor [21]]. When the suspension has a wholly isotropic structure, the effect of the presence of the particles is simply equivalent to an increase in the shear viscosity of the suspension. The magnitude of this increase, expressed as a fraction of the viscosity μ of the ambient fluid, is linear function of

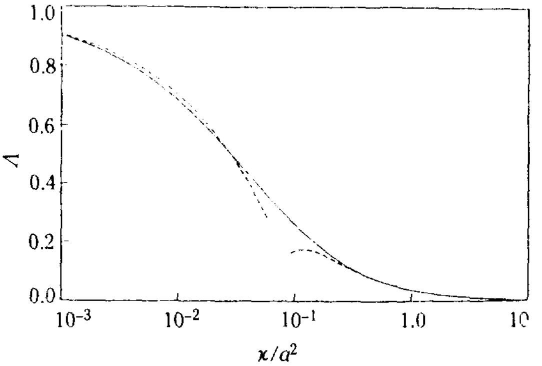

Fig. 5. Viscosity constant, Λ, as a function of the dimensionless permeability $\frac{\kappa}{a^2}$; ——, for the exact solution of (48); - - - -, for the asymptotic solutions of (49a,b).

the concentration of the particle by volume, and the constant of proportionality Λ depends on the constitution of the particles. In particular, for the case of a suspension of identical porous spheres, the viscosity constant Λ is given by

$$\Lambda = -\frac{1}{4} A^{o}_{2,0}(\sigma) \qquad (48)$$

which is the magnitude of the stresslet strength. When $\sigma \to \infty$ or $\sigma \to 0$, Kojima [13] evaluated asymptotically the viscosity constant Λ that can be readily reproduced by applying the asymptotic properties of spherical Bessel functions to $A^{o}_{2,0}(\sigma)$:

$$\Lambda = \frac{5}{2}(1 - 3\sigma^{-1}) + O(\sigma^{-2}), \quad as \ \sigma \to \infty \qquad (49a)$$

and

$$\Lambda = \frac{1}{10}\sigma^2 (1 - \frac{2}{35}\sigma^2) + O(\sigma^6), \quad \sigma \to 0. \qquad (49b)$$

In Figure 5, the viscosity constant Λ of (48) is plotted versus the dimensionless permeability. Also shown for comparison are the corresponding asymptotic solutions of (49a,b). It can be noted that the asymptotic forms provide an excellent approximation in the limit of large or small permeability.

It is worth commenting that the scope of the analysis can be readily extended to determine the particle motion in any general linear or quadratic flows in the presence of a plane interface. However, it should be mentioned that the undisturbed flow be compatible with the presence of the interface. In conclusion, we can also solve for an arbitrary motion of a particle in the presence of a *deformable* interface by means of a surface distribution of fundamental singularities (i.e., integral representation of fundamental solutions) for both Stokes and Brinkman equations.

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NOMENCLATURE

a	:	sphere radius
$A_{n,m}$, $D_{n,m}$	:	coefficients of spherical harmonic p_n of order n rank m
$B_{n,m}$, $E_{n,m}$	:	coefficients of spherical harmonic ϕ_n of order n rank m
$C_{n,m}$	:	coefficient of spherical harmonic χ_n of order n rank m
$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$	:	base vectors in the Cartesian coordinate system x,y,z
$\mathbf{F}$	:	hydrodynamic force
$I_{n+\frac{1}{2}}(\sigma r)$	:	modified Bessel function of order $n+\frac{1}{2}$
κ	:	permeability
K	:	flow parameter
l_c	:	characteristic length scale
$\mathbf{L}$	:	strain rate tensor
p	:	pressure field
p_c	:	characteristic stress scale
P_n, ϕ_n, χ_n	:	solid spherical harmonics of order n
$P^{\infty}(\mathbf{x})$	:	undisturbed pressure field
(r,θ,ϕ)	:	spherical coordinate system
$\mathbf{T}$	:	hydrodynamic torque
$\mathbf{u}$	:	velocity field
u_c	:	characteristic velocity scale
$\mathbf{u}_D$	:	velocity for potential doublet
$\mathbf{u}_{PQ}$	:	velocity for potential quadrupole
$\mathbf{u}_S$	:	velocity for Stokeslet
$\mathbf{u}_{SS}$	:	velocity for stresslet
$\mathbf{U}$	:	translational velocity
$\mathbf{U}^{\infty}(\mathbf{x})$	:	undisturbed velocity field
$\mathbf{x}$	:	position vector
$Y_{n,m}$	:	surface spherical harmonic of order n rank m
α	:	Stokeslet
β	:	potential doublet
γ	:	rotlet
Γ	:	shear rate
Λ	:	viscosity constant
μ	:	viscosity of fluid
Ω	:	angular velocity
σ	:	parameter defined by $\sigma^2 = \frac{a^2}{\kappa}$
τ	:	stress field
ξ	:	flow parameter

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