Korean J. of Chem. Eng., 4(2) (1987) *128-134*

THERMAL INSTABILITY IN A FLUID LAYER SUBJECTED TO TRANSIENT COOLING FROM ABOVE

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(Received 25 August 1986 • accepted 6 May 1987)

Abstract— The onset of natural convection in a horizontal fluid layer cooled from above is investigated theoretically. The initially quiescent fluid placed between two flat plates is cooled by decreasing the upper boundary temperature at a constant time-rate. Its stability analysis is conducted by employing the propagation theory, which considers variations of disturbances with the time upon their onset. The critical conditions predicted by this theory are found to agree favorably with the existing experimental results. Also, the effect of the Prandtl number on instability is discussed.

INTRODUCTION

The onset of natural convection in a horizontal fluid layer experiencing a sudden change in boundary temperature has been investigated extensively [1-6]. Most of previous studies have been conducted based on the quasi-static model and also on the linear amplification theory. The former model neglects the variations of disturbance quantities with time, while the latter one requires both an initial condition and its amplification factor. Limited agreement between the experimental results and theoretical predictions has been shown, but the individual model loses the validity to a certain degree. The quasi-static model leads to time-independent problems, which loses the effect of the Prandtl numbers on the criterion for the determination of the onset of convection time. In the amplification theory, the type of initial conditions and the amplification factor to correlate the theory with a particular experiment must be established. Therefore, the means to predict the critical conditions to mark the onset without the loss of generality is clouded in both models.

Recently Choi, Shin and Hwang [7] suggested the propagation theory that disturbances are initiated under the principle of exchange of stabilities but they do not experience conventional quasi-static characteristics. Under this theory disturbances will grow non-exponentially upon their initiations for a given large Rayleigh number. This new deterministic model has several advantages:

1. The effect of the Prandtl number on instability is obviously seen in stability equations.
2. The stability criteria are obtained directly from the stability equations.
3. This model does not need an initial condition or amplification ratio.

Choi et al. [7-8] applied this model to plane Couette flow and also plane Poiseuille flow with success.

The problem considered here is that of a horizontal fluid layer cooled from above by decreasing the temperature of the upper surface at a constant temporal rate. This is an extension of the work of Choi et al. [9], wherein the quasi-static approach was critically examined. The purpose of this investigation is to clarify the stability criteria by applying the propagation theory. In this regard, the base temperature profile is approximated by integral methods and simulation method which give the different penetration depth from one another.

MATHEMATICAL FORMULATION

The problem of interest is that of the Newtonian fluid layer confined by two rigid boundaries, as shown in Fig. 1. After introducing the Boussinesq approximation, the governing equations based on the linear stability theory can be decomposed into the unperturbed equations and the perturbation equations.

Unperturbed Equations

The unperturbed equation governing a fluid at rest can be expressed as the heat conduction equation, in a dimensionless form, which depicts a base temperature profile:

$$\frac{\partial \theta_0}{\partial \tau} = \frac{\partial^2 \theta_0}{\partial z^2} \tag{1}$$

with the initial and boundary conditions

$$\theta_0 (z, 0) = 0 \tag{2}$$

$$\theta_0 (0, \tau) = -\tau \text{ and } \theta_0 (1, \tau) = 0 \text{ for } \tau \geq 0 \tag{3}$$

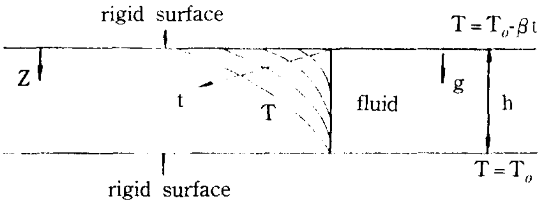

Fig. 1. A schematic diagram of the system.

A Leveque-type solution under the deep-pool approximation that the thermal penetration depth is very small in comparison with the whole depth of a fluid layer is obtained as follows:

$$\theta_0\,(z,\tau)=\frac{z\sqrt{\tau}}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4\tau}\right)-\left(\frac{z^2}{2}+\tau\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\tau}}\right) \tag{4}$$

But, the above solution accompanies mathematical difficulty in the stability analysis. Therefore, Choi et al. [9] approximated the base temperature profile by the following form:

$$\theta_0=-\tau\,(1-z/\delta_1)^{3.2687}\quad \text{for}\ 0\leqq z\leqq\delta_1 \tag{5}$$

$$\theta_0=0\quad \text{for}\ \delta_1\leqq z\leqq 1 \tag{6}$$

where δ_1 represents the thermal penetration depth ($\theta_0/\delta_1=-0.01$; $\delta_1=2.898\sqrt{\tau}$). For the simplification of mathematical treatment, the approximate solution is obtained by using the integral method:

$$\theta_0=-\tau\,(1-z/\delta_2)^{n}\quad \text{for}\ 0\leqq z\leqq\delta_2 \tag{7}$$

$$\theta_0=0\quad \text{for}\ \delta_2\leqq z\leqq 1 \tag{8}$$

The above equations satisfy the boundary conditions:

$$\theta_0=-\tau\ \text{and}\ \frac{\partial\theta_0}{\partial\tau}=-1\quad \text{at}\ z=0 \tag{9}$$

$$\theta_0=\frac{\partial\theta_0}{\partial z}=\frac{\partial^2\theta_0}{\partial z^2}=0\quad \text{at}\ z=\delta_2 \tag{10}$$

The energy equation produces the value of $\delta_2=\sqrt{20\tau}$ for n=5, and $\delta_2=\sqrt{8\tau}$ for n=3. All these solutions are compared each another, in a normalized form, in Fig. 2. It is found that the fifth-order polynomial having $\delta_2=\sqrt{20\tau}$ agrees well with the exact solution.

Perturbation Equations

The linearized perturbation equations are derived from the equations of continuity, momentum, and energy under the Boussinesq approximation as usual [9]. The resulting equations are obtained by considering that the disturbances show the characteristics of two-dimensional periodic waves near the onset time as follows:

$$\left\{\frac{1}{Pr}\frac{\partial}{\partial\tau}-\left(\frac{\partial^2}{\partial z^2}-a^2\right)\right\}\left(\frac{\partial^2}{\partial z^2}-a^2\right)w=a^2\theta \tag{11}$$

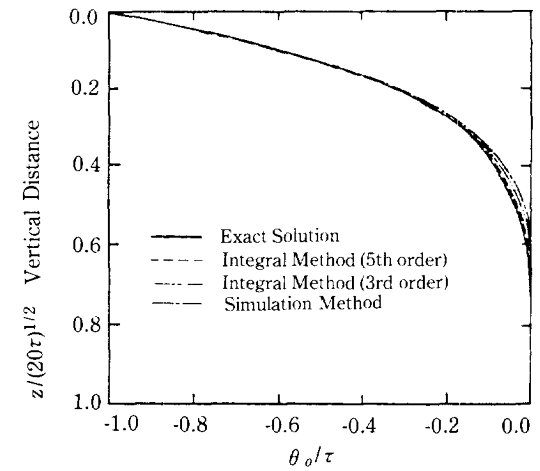

Fig. 2. Base temperature profiles in various methods.

$$\left\{\frac{\partial}{\partial\tau}-\left(\frac{\partial^2}{\partial z^2}-a^2\right)\right\}\theta=-Ra\frac{\partial\theta_0}{\partial z}w \tag{12}$$

From these equations the values of 'Ra' and 'a' must be found for given τ and Pr.

In a conventional stability analysis, the time was considered as a parameter so that time-dependence of disturbance quantities could be neglected in perturbation equations. In the present study, the propagation theory which considers the variations of disturbance quantities with time is considered. For the stability analysis the similarity variables are introduced as follows:

$$\zeta=g_2 z \tag{13}$$

$$w=g_1 w^*(\zeta)\ \text{and}\ \theta=\theta^*(\zeta) \tag{14}$$

where g_1 and g_2 are functions of the thermal penetration depth, which can be given as δ^2 and $1/\delta$ in this system, respectively. Perturbation equations are transformed by using the above similarity variables:

$$(D^2-a^{*2})\left\{(D^2-a^{*2})+\frac{A^2}{2Pr}(\zeta D-2)\right\}w^*=-a^{*2}\theta^* \tag{15}$$

$$\left(D^2+\frac{A^2}{2}\zeta D-a^{*2}\right)\theta^*=Ra^*(D\theta_0)w^* \tag{16}$$

where $D=d/d\zeta$, $A=\delta/\sqrt{\tau}$, $Ra^*=Ra\,\delta^3$ and $a^*=a\delta$. The resulting equations are valid at the onset time τ_c, when the parameters $Ra^*\tau$ and a^* are kept constant [11]. By substituting equation (16) into equation (15), the stability equations can be obtained as follows:

$$\left(D^2+\frac{A^2}{2}\zeta D-a^{*2}\right)\left[(D^2-a^{*2})^2+\frac{A^2}{2Pr}(\zeta D^3-a^{*2}\zeta D-2a^{*2})\right]w_t^*+Ra^*a^{*2}(D\theta_0)w_t^*=0$$

for $0 \leqq \zeta \leqq 1$ (17)

$$(D^2+\frac{A^2}{2}\zeta D-a^{*2})[(D^2-a^{*2})^2+\frac{A^2}{2Pr}(\zeta D^3-a^{*2}\zeta D+2a^{*2})]w_o^*=0$$

for $\zeta \geqq 1$ (18)

where the subscripts i and o indicate the inside and the outside region of the thermal penetration depth, respectively.

SOLUTION PROCEDURE

For infinite-Pr fluids, the convective term in stability equations can be ignored. Therefore, the resulting equations are expressed as follows:

$$[(D^2+\frac{A^2}{2}\zeta D-a^{*2})(D^2-a^{*2})^2+Ra^*a^{*2}(D\theta_0)]w_i^*=0 \qquad \text{for } 0\leqq\zeta\leqq1 \qquad (19)$$

$$(D^2+\frac{A^2}{2}\zeta D-a^{*2})(D^2-a^{*2})^2w_o^*=0 \qquad \text{for } \zeta\geqq1 \qquad (20)$$

Boundary conditions for a deep-pool system become

$$w^*=Dw^*=0 \qquad \text{at } \zeta=0 \text{ and } \zeta\to\infty \qquad (21)$$

Because the temperature disturbances vanish at a rigid boundary, another boundary condition can be obtained from equation (15):

$$(D^2-a^{*2})^2w^*=0 \qquad \text{at } \zeta=0 \text{ and } \zeta\to\infty \qquad (22)$$

Now, two solutions of equations(19)and (20), which satisfy the boundary conditions, will be patched each other at $\zeta=1$. Interface conditions at $\zeta=1$ are given by continuity of velocity, momentum, and stress:

$$w_i^*=w_o^* \text{ and } D^nw_i^*=D^nw_o^* \ (n=1,2,\cdots,5) \qquad (23)$$

Equations (19) and (20) can be analytically solved by following the method of Choi et al. [7].

A solution of equation (19) can be easily determined in a power series form as follows:

$$w_i^*=\sum_{l=1}^{5}H_l f^{(l)}(\zeta) \qquad (24)$$

$$f^{(l)}(\zeta)=\sum_{n=0}^{\infty}b_n^{(l)}\zeta^n \qquad (25)$$

where H_l is an arbitrary constant and $b_n^{(l)}$ can be constructed to satisfy equation (19) with the base temperature profile by the integral method. For example, when the base temperature profile is approximated as a fifth order polynomial, $b_n^{(l)}$ becomes

$$b_n^{(l)}=\frac{1}{n!}[3a^{*2}(n-2)!b_{n-2}^{(l)}-3a^{*2}(n-4)!b_{n-4}^{(l)}+(n-6)!((a^{*6}-5Ra^*\tau a^{*2})b_{n-6}^{(l)}-20Ra^*\tau a^{*2}b_{n-7}^{(l)}-30Ra^*\tau a^{*2}b_{n-8}^{(l)}+20Ra^*\tau a^{*2}b_{n-9}^{(l)}-5Ra^*\tau a^{*2}b_{n-10}^{(l)})]$$

for $n\geqq6$ (26)

where $b_n^{(l)}=0$ for $n\leqq0$,

$b_n^{(l)}=\delta_{nl}$ for $0\leqq n\leqq5$.

This study is concerned only with the case of very small time marking the onset of thermal convection. So, the general solution of equation (20) should be obtained in the infinite domain and this brings complication in its mathematical treatment. Therefore, the equation (20) can be separated as follows:

$$(D^2+\frac{A^2}{2}\zeta D-a^{*2})Y=0 \qquad (27)$$

$$(D^2-a^{*2})^2w_o^*=Y \qquad (28)$$

After transforming a coordinate in equation (27) to $s=\zeta-1$, its solution is obtained in exponential forms. Also, we can find the asymptotic solution of equation (27) which satisfies the boundary condition (22) as $\zeta\to\infty$ by adopting the WKB approximation method:

$$Y(\zeta)\cong\exp[-A^2\zeta^2/8-\int_1^{\zeta}(\frac{A^4}{16}x^2+\frac{A^4}{4}+a^{*2})^{1/2}dx]/(\frac{A^4}{16}\zeta^2+\frac{A^2}{4}+a^{*2})^{1/4} \qquad (29)$$

The above solution provides the boundary conditions at $s=0$ for the exponential forms obtained above. Sequently w_o^* in equation (28) can be easily obtained by an operator technique:

$$w_o^*=(H_6+H_7s)e^{-a^*s}+H_8[e^{a^*s}P(s)+e^{-a^*s}Q(s)]/4a^{*2} \qquad (30)$$

where

$$P(s)=\sum_{n=0}^{\infty}\frac{P_n}{(n+1)(n+2)}s^{n+2}-\frac{1}{a^*}\sum_{n=0}^{\infty}\frac{P_n}{(n+1)}s^{n+1}$$

$$P_n=-[(2a^{*2}+\frac{A^2}{2})(n-1)P_{n-1}+\frac{A^2}{2}(n-2+a^{*2})P_{n-2}+\frac{A^2}{2}a^*P_{n-3}]/n(n-1)$$

$$P_{-1}=0,\quad P_0=Y(1),\quad P_1=Y'(1)-Y(1)$$

$$Q(s)=\sum_{n=0}^{\infty}\frac{q_n}{(n+1)(n+2)}s^{n+2}+\frac{1}{a^*}\sum_{n=0}^{\infty}\frac{q_n}{(n+1)}s^{n+1}$$

$$q_n=[(2a^{*2}-\frac{A^2}{2})(n-1)q_{n-1}+\frac{A^2}{2}(a^*-n+2)q_{n-2}+\frac{A^2}{2}a^*q_{n-3}]/n(n-1)$$

$$q_{-1}=0,\quad q_0=Y(1),\quad q_1=Y'(1)+Y(1)$$

Finally, the critical conditions can be determined from the following secular equations which are obtained by applying the conditions (21) and (22):

$$\begin{vmatrix} f^{(2)}+\frac{a^*}{6}f^{(4)} & f^{(3)} & f^{(5)} & -1 & 0 & 0 \\ f^{(2)\prime}+\frac{a^*}{6}f^{(4)\prime} & f^{(3)\prime} & f^{(5)\prime} & a^* & -1 & 0 \\ f^{(2)\prime\prime}+\frac{a^*}{6}f^{(4)\prime\prime} & f^{(3)\prime\prime} & f^{(5)\prime\prime} & -a^{*2} & 2a^* & 0 \\ f^{(2)\prime\prime\prime}+\frac{a^*}{6}f^{(4)\prime\prime\prime} & f^{(3)\prime\prime\prime} & f^{(5)\prime\prime\prime} & a^{*3} & -3a^{*2} & 0 \\ f^{(2)IV}+\frac{a^*}{6}f^{(4)IV} & f^{(3)IV} & f^{(5)IV} & -a^{*4} & 4a^{*3} & -Y(1) \\ f^{(2)V}+\frac{a^*}{6}f^{(4)V} & f^{(3)V} & f^{(5)V} & a^{*5} & -5a^{*4} & -Y'(1) \end{vmatrix} = 0 \quad (31)$$

where the prime represents differentiation with respect to ζ . In this system, $Ra^*\tau$ is expressed as an eigenvalue for a given wave number, a^*. Fig. 3 shows a typical stability diagram for determining the critical condition. All of disturbances do not grow until $Ra^*\tau$ reaches $(Ra^*\tau)_c$, while disturbances will grow for $Ra^*\tau > (Ra^*\tau)_c$.

In case of very small Prandtl numbers, by looking back at equations (17) to (18), new stability equations can be reformulated. But, it can be assumed that, for small Prandtl numbers, velocity disturbances are confined within the thermal boundary layer by scale analysis [10]. Also, by introducing Choi's modified concept [4] that "temperature disturbances at the onset of natural convection are confined within the thermal boundary layer thickness", the stability equation is easily formulated as follows:

$$\left[\frac{A^2}{2}\left(D^2+\frac{A^2}{2}\zeta D-a^{*2}\right)\left(\zeta D^3-a^{*2}\zeta D+2a^{*2}\right) + Pr\ Ra^* a^{*2}(D\ \theta_0)\right]w_i^* = 0 \quad \text{for } 0\leq\zeta\leq 1 \quad (32)$$

$$w_o^* = 0 \qquad \text{for } \zeta \geq 1 \quad (33)$$

The above equation makes us be able to observe the qualitative charateristics of the instability in very small-Pr fluids without mathematical difficulty. It is important to note that $PrRa^*\tau$ for very small-Pr fluids plays the same role as $Ra^*\tau$ for infinite-Pr fluids. The stability criteria can be obtained by the same procedures as the case of the infinite Prandtl number. But, the boundary conditions must be relaxed under the approximation for very small-Pr fluids so that the no slip condition can not be applicable. The boundary conditions become

$$w_i^* = 0 \text{ and } \theta^* = 0 \qquad \text{at } \zeta = 0 \quad (34)$$

$$w_i^* = 0,\ Dw_i^* = 0 \text{ and } \theta^* = 0 \qquad \text{at } \zeta \leftarrow 1 \quad (35)$$

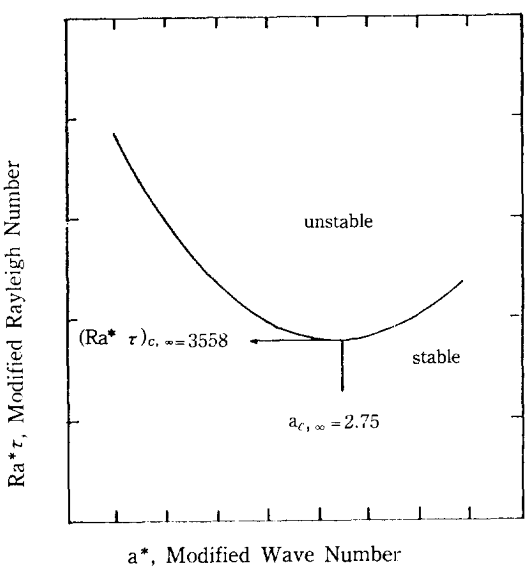

Fig. 3. A typical stability diagram for infinite Prandtl number.

RESULTS AND DISCUSSION

The theoretical results of this study are presented for the infinite Prandtl number in Table 1. Integral methods (1) and (2) mean that they are approximated as a fifth-and a third-order polynomial, respectively.

These critical values may be modified by using the relationship between the thermal penetration depth and time. It enables us to compare theoretical values with experimental data of Davenport and King [3] as shown in Fig. 4.

The critical conditions based on the propagation theory are lower than experimental values, which may reflect the growth time required for a finite disturbance to be observed and also the influence of Prandtl numbers on the critical conditions. The critical time is inversely proportional to 2/5 power of the calculated Rayleigh number in accordance with the experimental observation.

In the present propagation theory, the thermal penetration depth is used as a length scaling factor.

Table 1. Critical conditions with $\tau_c = m\ Ra^{-2/5}$ and $a_c = n\ Ra^{1/5}$ for $Pr \to \infty$.

Base Temperature Profile	$Ra^*\tau$	a^*	m	n
Integral Method (1)	3560	2.75	4.36	0.29
Integral Method (2)	1063	1.99	4.64	0.31
Simulation Method	1214	2.00	4.78	0.32

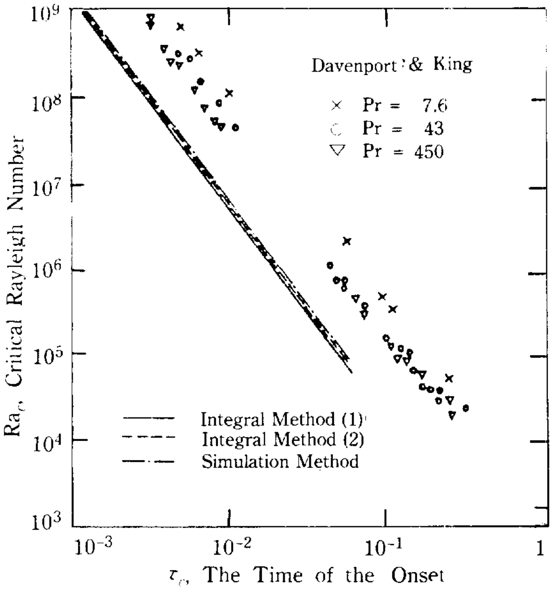

Fig. 4. The time of the onset of convection vs. critical Rayleigh number.

Even though three different values are used, the resulting critical conditions in the form of $\tau_c = mRa^{-2/5}$ are almost the same as summarized in Table 1. Therefore, it may be stated that the simple approximation to describe the basic temperature profile can be used for the stability analysis. The distributions of disturbance amplitudes are illustrated in normalized forms in Fig. 5. From this figure the value of the thermal penetration depth influences the pattern of disturbance distribution to a certain degree.

In this connection, it is necessary to carefully examine the base temperature profiles in Fig. 2. The differences between the approximate profile and the exact one produce a little different distribution between amplitude functions. But they are negligible in predicting the critical values. This is the reason why a number of trial functions have generated the critical conditions with success [1,2,6]. As of now, the following stability criteria based on the fifth order polynomial is the most precise, since it is the closest to the exact temperature distribution:

$$\tau_c = 4.36\ Ra^{-2/5} \text{ and } a_c = 0.29\ Ra^{1/5} \quad \text{for } Pr \to \infty \qquad (36)$$

Kaviany [6] reported that the amplification theory produced the relation of $\tau_c = 18.6Ra^{-2/5}$ for Pr=7 and $\tau_c = 19.7Ra^{-2/5}$ for Pr→∞. His critical time is the time at which the magnitude of the disturbance grows by one thousand times its initial value. His results for Pr=17 and the prediction for Pr→∞ are compared in

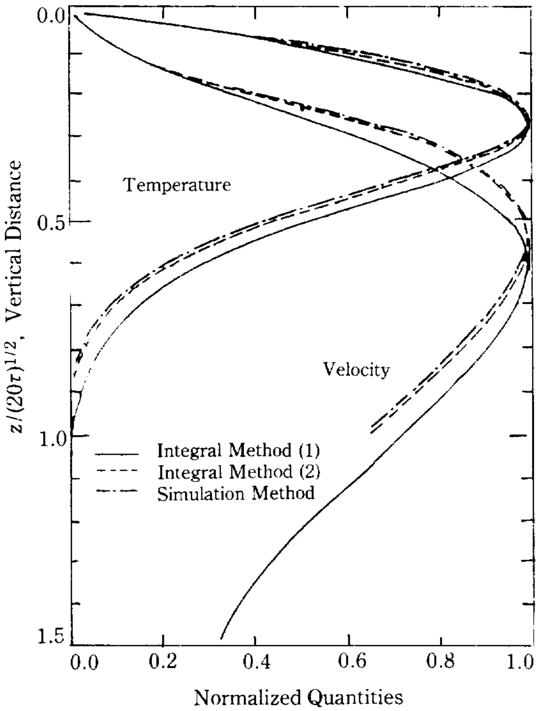

Fig. 5. The distributions of amplitude functions for infinite Prandtl number.

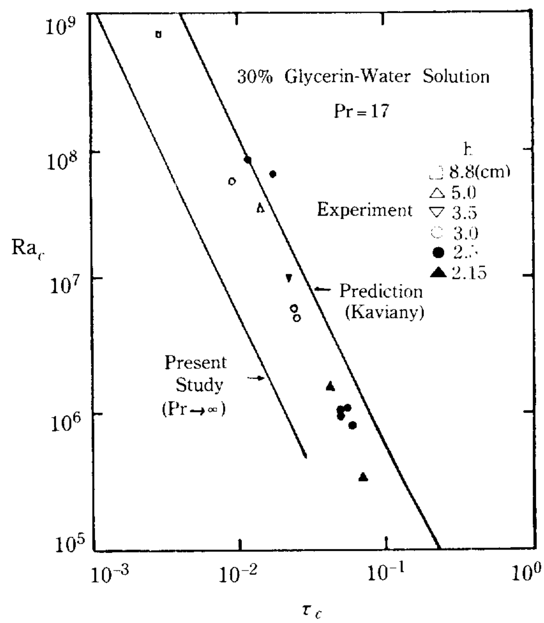

Fig. 6. The comparison between the present study and the experimental results of Kaviany [6].

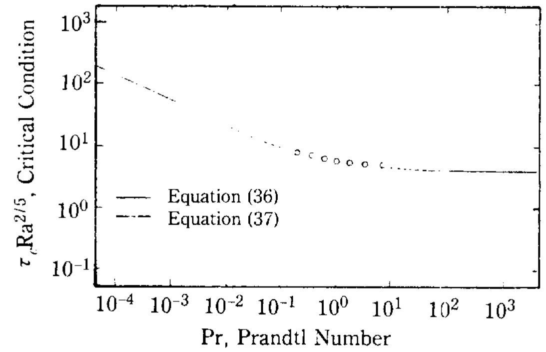

Fig. 7. The effect of Prandtl numbers on the stability criteria under the assumptions of $w_o^*=0$ and $\theta_o^*=0$.

Fig. 6. From this figure the present prediction is found more reliable, since the time period when disturbances grow to a discernible size is required.

The critical time predicted by equation (36) is a little higher than that of Choi et al.'s work [9]. They used the modified quasi-static model that the temperature disturbances are confined within the thermal penetration depth. This assumption is found quite reasonable from Fig. 5. Therefore this concept may be extended to low-Pr cases. For Pr<1, it may be probable to confine both velocity and temperature disturbances within the thermal penetration depth. This kind of treatment removes mathematical complication. It is found that the lower the Prandtl number is, the more stable the system is. Therefore the stability criteria show the trend as illustrated in Fig. 7. In the case of Pr → 0, the critical conditions are represented by

$$\tau_c = 3.32\,(\mathrm{Pr\ Ra})^{-2/5} \text{ ane } a_c = 0.577\,(\mathrm{PrRa})^{1/5} \tag{37}$$

It is clear that for a given Rayleigh number the critical time increases as the Prandtl number decreases. This means that the system becomes more stable.

CONCLUSION

Predicted results of the time of the onset of thermal convection for a horizontal fluid layer cooled from above have been presented. The propagation theory predicts very favorable critical conditions. Through comparison with existing results it is found that the present theory is quite reasonable. Also, it seems evident that the temperature disturbances are initiated within the thermal penetration depth.

The results presented here complement the work of Choi, Yeo, Kwon and Yoo [9].

ACKNOWLEDGEMENT

The authors acknowledge the financial support of the Korea Science & Engineering Foundation, Daedeok, Korea.

NOMENCLATURE

A^2	:	constant in equation (15), δ^2/τ
a	:	wave number
a*	:	modified wave number, $a\delta$
a_n, c_n	:	coefficients in equation (29)
D	:	differential operator with respect to ζ
g	:	gravitational acceleration, m/sec
H_i	:	constants in solutions
h	:	layer thickness, m
Pr	:	Prandtl number, ν/κ
Ra	:	Rayleigh number, $\alpha\beta g h^5/\kappa^2\nu$
Ra*	:	modified Rayleigh number, $Ra\delta^3$
s	:	variable, ζ -1
T	:	temperature, K
t	:	time, sec
W	:	vertical component of velocity perturbation
w	:	dimensionless velocity, Wh/κ
w*	:	modified velocity, w/δ^2
Z	:	vertical distance, m
z	:	dimensionless vertical distance, Z/h

Greek Letters

α	:	thermal expansivity, 1/K
β	:	temporal rate in temperature rate
δ	:	dimensionless thermal penetration depth
δ_{ni}	:	Kronecker delta
ζ	:	modified vertical distance, z/δ
θ, θ^*	:	temperature perturbation non-dimensionalized by $\alpha g h^3/\nu\kappa$
θ_0	:	base temperature non-dimensionalized by $\kappa/\beta h^2$
κ	:	thermal diffusivity, m^2/sec
ν	:	kinematic viscosity, m^2/sec
τ	:	time non-dimensionalized by κ/h^2

Subscripts

c	:	critical state
i	:	region for $\zeta \leqq 1$
o	:	region for $\zeta \geqq 1$
0	:	base state

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