Korean J. Chem. Eng., 33(2), 416-422 (2016)
DOI: 10.1007/s11814-015-0153-4

pISSN: 0256-1115
eISSN: 1975-7220

Globally stable control systems for processes with input multiplicities

Jietae Lee*,† and Thomas F. Edgar**

*Department of Chemical Engineering, Kyungpook National University, Daegu 702-701, Korea
**Department of Chemical Engineering, University of Texas, Austin, TX78712, U.S.A.
(*Received 16 February 2015 • accepted 10 July 2015*)

Abstract−A nonlinear process with input multiplicity has two or more input values for a given output at the steady state, and the process steady state gain changes its sign as the operating point changes. A control system with integral action will be unstable when both signs of the process gain and the controller integral gain are different, and its stability region will be limited to the boundary where the process steady state gain is zero. Unlike processes with output multiplicities, feedback controllers cannot be used to correct the sign changes of process gain. To remove such stability limitation, a simple control system with parallel compensator is proposed. The parallel compensator can be easily designed based on the process steady state gain information and tuned in the field. Using the two time scale method, the stability of proposed control systems for processes with input multiplicities can be checked.

Keywords: Input Multiplicity, PI Control, Parallel Compensator, Two Time Scale Analysis, Global Stability

INTRODUCTION

Chemical processes with complex behaviors such as high nonlinearity, non-minimum phase, instability, and steady state input and output multiplicities present troublesome control problems. Although each poses its own operational difficulty, a steady state input multiplicity is known to cause one of the most difficult control problems [1]. A process with input multiplicity has two or more input values for a given output at the steady state. This implies that the process steady state gain changes its sign as the operating point changes and a linear controller with integral action has a limited convergence region where both the process gain and the controller integral gain have the same sign [2,3]. Nonlinear control systems even with complicated models [4,5] can suffer from instability. Various processes show input multiplicities, for example, chemical reactors [6], distillations [7,8], and recycle processes [9]. For some processes, large disturbances can change the sign of the process steady state gain, causing instability of control systems that have integral actions for offset-free operation [10].

Input multiplicities accompanied by non-minimum phase dynamics [3] are considered to be what should be avoided in the process design step. The continuation method is often used to find whether input multiplicities appear [11,12] and operability analyses are based on this [13,14]. For some processes, such troublesome behaviors can be avoided by slight changes in the process design.

A process with input multiplicity has two or more input values for a given output. Hence, when such input values are sufficiently apart from each other; the given operating point can be controlled by imposing constraints on the input [7,15,16]. For a multi-input multi-output process, control constraints can be imposed on the control direction [9]. A multiple model approach can also be applied to control processes with input multiplicities [17,18]. In both approaches, models should be accurate enough to approximate the process gain changes accurately.

In this study, we propose a simple control system for processes with input multiplicities. A parallel compensator is introduced to cope with the sign changes in the process steady state gain. It enlarges the convergence region effectively. A method to avoid offsets that occur due to the parallel compensator is provided. The stability of proposed control systems for processes with input multiplicities can be checked via the two time scale method [19].

PARALLEL COMPENSATOR

Consider a stable nonlinear dynamic process

$$\dot{x}(t)=f(x(t), u(t)) \quad y(t)=h(x(t)) \tag{1}$$

where x(t) is an n dimensional state vector, u(t) and y(t) are scalar input and output variables, respectively. Let, for a given constant input $u(t)=u_{ss}$, the steady state value satisfying f(x, u)=0 be

$$x_{ss}\equiv\xi(u_{ss}) \tag{2}$$

Then the static relationship between input and output is

$$y_{ss}=h(x_{ss})=h(\xi(u_{ss}))\equiv q(u_{ss}) \tag{3}$$

The process steady state gain is

$$\frac{dy_{ss}}{du_{ss}}=q'(u_{ss}) \tag{4}$$

For a process with input multiplicity, its sign changes as the input u_{ss} varies. A control system with integral action can be unstable as the sign of process steady state gain changes. For model-based control systems, process models predicting the sign change of steady

†To whom correspondence should be addressed.
E-mail: jtlee@knu.ac.kr

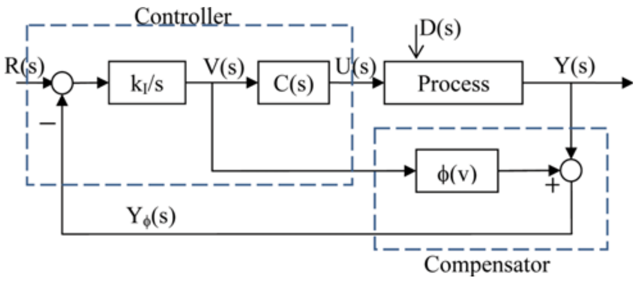

Fig. 1. Control system with a parallel compensator for the process with input multiplicity.

state gains accurately are required [17,18]. Otherwise, to estimate process steady state gains adaptively, complicated schemes should be equipped. They often require continuous perturbations for convergence.

To overcome difficulties due to the input multiplicities, a parallel compensator as shown in Fig. 1 is considered. Here a linear integral controller whose gain is k_I and C(s) representing the proportional and high order parts (C(0)=1) are used to control the process. The feedback signal is now

$$y_\phi(t)=y(t)+\phi(v(t)) \tag{5}$$

At the steady state, $v_{ss}=u_{ss}$. The parallel compensator is applied only to the integral controller output because the static compensator ϕ(.) shows bad effects such as chattering for the proportional part of C(s). For faster closed-loop responses, C(s) compensating process dynamics is required. In addition, instead of constant k_I and linear controller C(s), nonlinear integral gain and nonlinear high order controller may improve closed-loop responses [20]. However, because our concern here is to compensate the static nonlinearity due to the sign change of process steady state gain, a control system without C(s) is considered (C(s)=1).

When a parallel compensator is introduced, the process steady state gain becomes

$$\frac{dy_{\phi_{ss}}}{du_{ss}}=q'(u_{ss})+\phi'(u_{ss}) \tag{6}$$

As shown in Fig. 2, we choose ϕ(u) such that Eq. (6) has the same sign throughout the whole input region. With such ϕ(u), control difficulties due to the input multiplicities can be removed in the feedback loop and integral controller can be applied without worrying about instability due to the sign change of process steady state gain.

Here, the set point should be

$$r=y_{ss}+\phi(u_{ss}) \tag{7}$$

When $\phi(u_{ss})$ is not zero, there exist offsets for sustained disturbances and process changes. The offsets are drawbacks caused by the parallel compensator. The left hand side in Fig. 2(a) is the offset-free region and the right hand side is the region where offset exists. For offset-free operation in the right hand side in Fig. 2(a), we may employ a different compensator as shown in Fig. 2(b). By switching the compensators and the controller gains, we can obtain control systems which are offset-free for both operating regions.

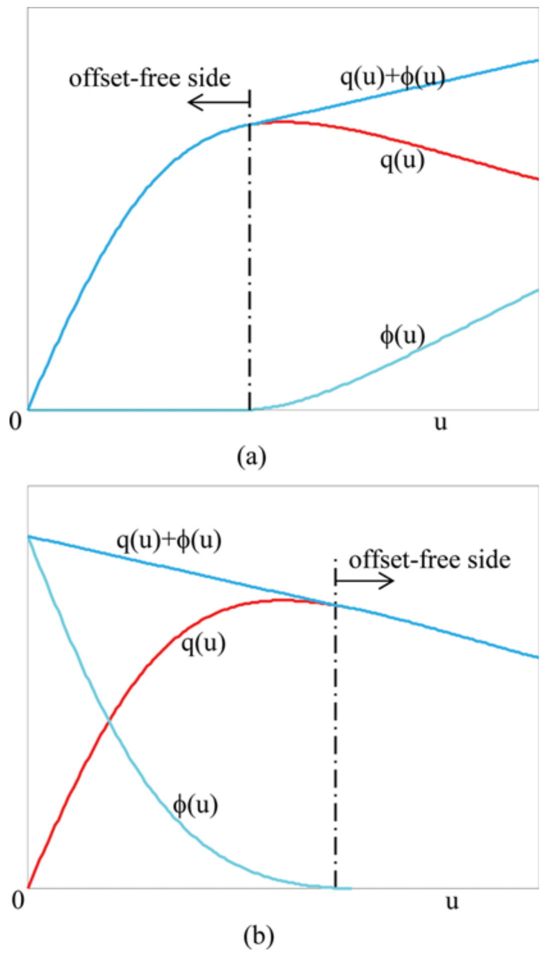

Fig. 2. Typical functions of q(u), ϕ(u) and q(u)+ϕ(u).

In summary, to compensate the sign change of process steady state gain, we use the parallel compensator ϕ(u) such that

(1) q′(u)+ϕ′(u) has the same sign throughout the whole range of input u.

(2) ϕ(u)=0, if possible, for u in the main operating region.

When the steady state relationship q(u) is poorly known, we use a simplified ϕ(u) having two tuning parameters, for q(u) as shown in Fig. 2(a) (without loss of generality, it is assumed that the left hand side in the plot q(u) is the main operating region and its slope is positive),

$$\phi(u)=\begin{cases} 0, & u<\phi_1 \\ \phi_2(u-\phi_1), & \text{otherwise} \end{cases} \tag{8}$$

Here, ϕ_1 is the input value where the process steady state gain is near zero and ϕ_2 is a gain greater than the largest absolute of negative process gains.

STABILITY ANALYSIS BASED ON THE TWO TIME SCALE METHOD

Consider an integral control system as shown in Fig. 1. It is assumed that the open loop system of Eq. (1) is uniformly stable and there exists a quadratic Lyapunov function. The closed loop system without C(s) is ($k_I=\varepsilon\tilde{k}_I$)

$$\begin{aligned}&\dot{x}(t)=f(x(t),\, u(t))\\&\dot{u}(t)=\varepsilon\tilde{k}_I(r-(h(x(t))+\phi(u(t))))\end{aligned}\tag{9}$$

Changing the time scale by $\tau=\varepsilon t$, we have

$$\begin{aligned}&\varepsilon\dot{x}(\tau)=f(x(\tau),\, u(\tau))\\&\dot{u}(\tau)=\tilde{k}_I(r-(h(x(\tau))+\phi(u(\tau))))\end{aligned}\tag{10}$$

The two time scale method [19] in the Appendix can now be applied to prove the stability of the closed-loop system. It requires finding Lyapunov functions for fast and slow subsystems of Eq. (10) and its composite Lyapunov function. Detailed Lyapunov functions are dependent on the system functions of f(x, u) and h(x).

The slow subsystem in Eq. (10) is

$$\begin{aligned}&0=f(x(\tau),\, u(\tau))\\&\dot{u}(\tau)=\tilde{k}_I(r-(h(x(\tau))+\phi(u(\tau))))\end{aligned}\tag{11}$$

and hence

$$\begin{aligned}\dot{u}(\tau)&=\tilde{k}_I(r-(h(\xi(u(\tau)))+\phi(u(\tau))))\\&=\tilde{k}_I(r-(q(u(\tau))+\phi(u(\tau))))\end{aligned}\tag{12}$$

When $\phi(u)$ is chosen so that $q'(u)+\phi'(u)$ has the same sign for the whole range of u, the system (12) has a simple Lyapunov function of $V(u)=(u-u_{ss})^2$. Whenever the sign of $q'(u)+\phi'(u)$ remains unchanged for process variations and disturbances, the proposed control system will exhibit robust stability. Without $\phi(u)$, for the process with input multiplicity, a valid Lyapunov function for the whole range of u does not exist and the stability region will be limited.

For some processes, the dissipativity based method [13] is a better choice to ensure stability of the proposed control system.

SIMULATIONS

There have been several methods to control processes with input multiplicities. A series compensator method [15,16] uses the Hammerstein-type nonlinear model and inversion of the model separately for positive and negative steady state gain regions. The method is very simple but has robustness problems for the changes of process nonlinearities. A multiple model method [17,18] uses multiple models to describe the steady state gain changes and model-based control with different or mixed models according to operating regions. This method still has robustness problems and can be complex to describe changes of process nonlinearities accurately. The proposed method is as simple as the series compensator method [15] and has robust stability for wide variations of process nonlinearities for which the sign of $q'(u)+\phi'(u)$ remains unchanged. Following simulations will illustrate these.

Example 1 : Consider a nonlinear process

$$\begin{aligned}&\dot{x}(t)=-x(t)+u(t-\theta)\\&y(t)=\frac{2x(t)}{1+x^2(t)}\end{aligned}\tag{13}$$

As shown in Fig. 3, it exhibits input multiplicity.

When $\theta=0$ (delay-free process), a stable control system with a

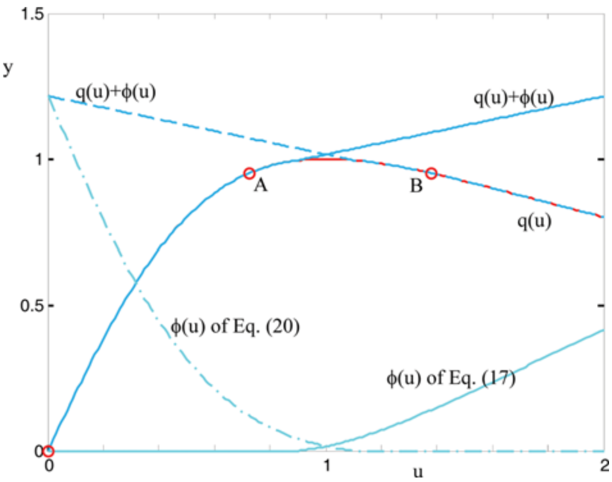

Fig. 3. Functions of q(u), ϕ(u) and q(u)+ϕ(u) for example 1 process.

parallel compensator can be easily obtained. Consider an integral control system

$$\begin{aligned}&\dot{x}(t)=-x(t)+u(t)\\&\dot{u}(t)=k_I\left(r-\frac{2x(t)}{1+x^2(t)}-\phi(u(t))\right)\end{aligned}\tag{14}$$

Introducing variables of $\tau=\varepsilon t$, $\tilde{x}=x-u$, $k_I=\varepsilon\tilde{k}_I$ we have

$$\begin{aligned}&\varepsilon\dot{\tilde{x}}(\tau)=-\tilde{x}(\tau)+\varepsilon\dot{u}(\tau)\\&\dot{u}(\tau)=\tilde{k}_I\left(r-\frac{2(\tilde{x}(\tau)+u(\tau))}{1+(\tilde{x}(\tau)+u(\tau))^2}-\phi(u(\tau))\right)\end{aligned}\tag{15}$$

When $\phi(u)$ is designed as, for positive constants c_1 and c_2,

$$c_1\le\frac{d}{du}\left(\frac{2u}{1+u^2}+\phi(u)\right)\le c_2\tag{16}$$

simple quadratic functions of

$$\begin{aligned}&W(\tilde{x},\, u)=|\tilde{x}|^2\\&V(u)=|u-u_{ss}|^2\end{aligned}\tag{17}$$

become Lyapunov functions for fast and slow subsystems of Eq. (15) satisfying all the conditions for stability in the Appendix. Without $\phi(u)$, we cannot find positive constants, c_1 and c_2, that satisfy Eq. (16) and the stability of closed-loop system cannot be realized.

For u_{ss} between 0 and 0.9, a parallel compensator satisfying Eq. (16) can be chosen as

$$\phi(u)=\begin{cases}0, & u<0.9\\ 0.9945-\dfrac{2u}{1+u^2}+0.2(u-0.9), & \text{otherwise}\end{cases}\tag{18}$$

Stability of the control system (14) with the parallel compensator (18) can be shown also by phase plane trajectories in Fig. 4. Here, we use r=0.95 and k_I=1.2. We can see that the proposed control system with the parallel compensator (18) is stable and all trajectories go to the point x=u=0.72395 such that $2x/(1+x^2)=r=0.95$. However, for the integral control system without the parallel compensator, some trajectories diverge.

For a nonzero θ, the stability proof is not simple. Properties of

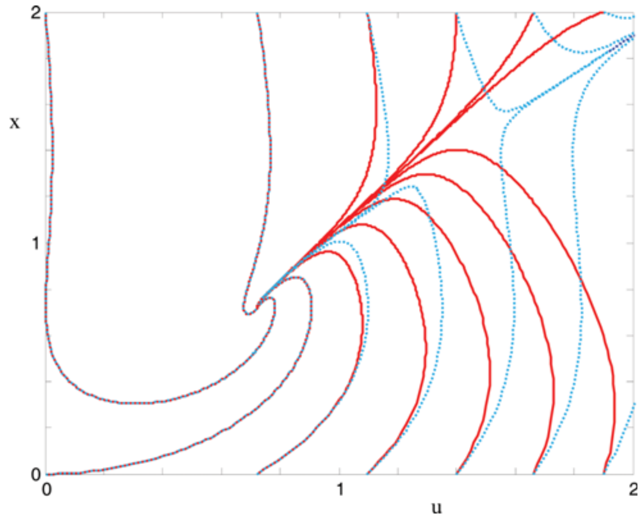

Fig. 4. Phase plane trajectories of the integral control systems for example 1 process without (dotted line) and with (solid line) the parallel compensator of Eq. (18) (r=0.95, k_I=1, θ=0).

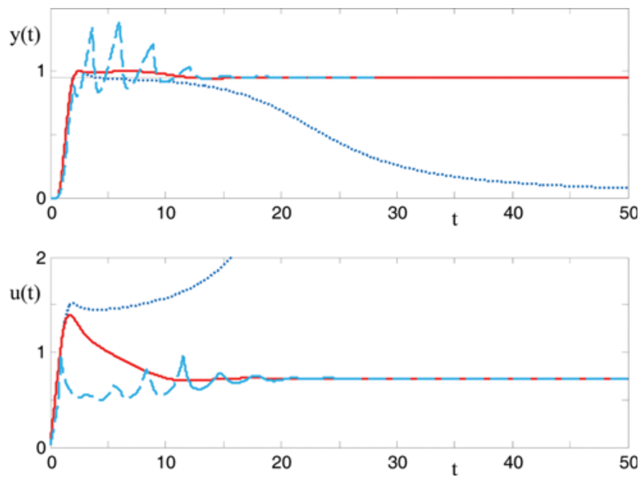

Fig. 5. Responses of the linear integral control (dotted line), the proposed integral control with parallel compensator (solid line), and the integral control with series compensator (dashed line; Chidambaram and Reddy [15]) for example 1 process (r=0.95, k_I=1.2, θ=0.6).

the proposed control system with parallel compensator are illustrated by simulations. Fig. 5 shows the closed-loop performance for r=0.95, θ=0.6 and k_I=1.2 with and without the parallel compensator of Eq. (18). The sampling time is set to 0.01. We can see that the integral controller without the parallel compensator fails to steer the process to the set point r=0.95. On the other hand, the proposed control system with parallel compensator succeeds. To cope with the input multiplicity, Chidambaram and Reddy [15] use a series compensator by approximating the process to be a Hammerstein-type nonlinear model. For this process, the series compensator is (the smaller root of $v=2u/(1+u^2)$)

$$u(t)=\frac{1-\sqrt{1-v(t)^2}}{v(t)} \tag{19}$$

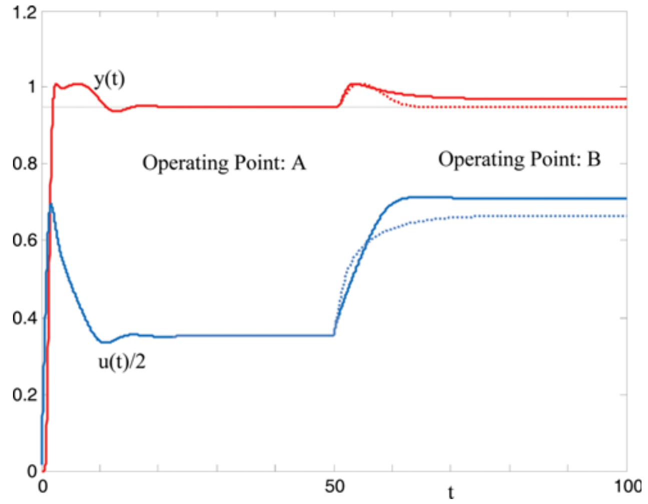

Fig. 6. Responses of the proposed integral control with parallel compensator for example 1 process (dotted line: the compensator and the integral gain are switched at t=50 to Eq. (20) and −1.2, respectively).

where v(t) is the output of integral controller. Control performance for k_I=1.2 is shown in Fig. 5. Smooth control performance can be obtained with decreased integral gain, but responses become more sluggish. For higher integral gain, the integral control system with the above series compensator fails to steer the process to the given operating point of r=0.95 due to the complex number in u(t) of Eq. (19). The controller output v(t) can be often greater than unity and this series compensator of Eq. (19) cannot work.

Fig. 6 shows the closed-loop performance of integral control systems with parallel compensators. Here, to show the effects of process variations, the output equation is changed a little as

$$y(t)=\frac{2.02x(t)}{1+x^2(t)} \tag{20}$$

With the parallel compensator (18), the proposed control system is applied for both operating points A and B in Fig. 3. For the operating point B, we use the set point of r=0.95+ϕ(1.3813)=1.0907. At the operating point B, the proposed control system with the parallel compensator (18) shows offsets due to the process parameter change of Eq. (20). For the offset-free operation at B, a different compensator should be used. For this, we use

$$\phi(u)=\begin{cases} 0.9955-\dfrac{2u}{1+u^2}-0.2(u-1.1), & u<1.1 \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

This compensator is designed for the process of Eq. (13). Fig. 3 shows q(u) and ϕ(u) for both Eqs. (18) and (21). Dotted lines in Fig. 6 show control performance when the parallel compensator is switched from Eq. (18) to Eq. (21) at t=50 and the integral gain k_I is switched from 1.2 to −1.2. The velocity form for the integral control is used for the bumpless transfer. We can see that the operating point of A is steered to B smoothly without offsets.

When a proportional controller of C(s)=s+1 is included instead of C(s)=1, faster closed-loop responses can be obtained.

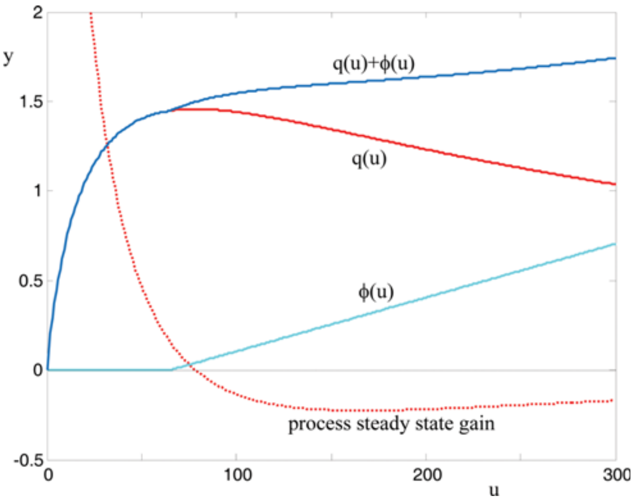

Fig. 7. Functions of q(u), ϕ(u) and q(u)+ϕ(u) for example 2 process (Van de Vusse reactor).

Example 2 (Van de Vusse Reactor): Consider the reactor process known as the Van de Vusse reactor [3].

$$\dot{x}_1(t)=-k_1x_1(t)-k_3x_1^2(t)+(x_{1f}-x_1(t))u(t)$$
$$\dot{x}_2(t)=k_1x_1(t)-k_2x_2(t)-x_2(t)u(t) \quad (22)$$
$$y(t)=x_2(t)$$

The steady state relationship between input and output is shown in Fig. 7. The kinetic parameters used are k_1=50, k_2=100, k_3=10.1, and x_{1f}=10.0. As shown in Fig. 7, the process shows input multiplicity. The process gain changes from 0.1045 to −0.0023 as the process input changes from 0 to 300. To compensate for the negative process gain, we design a parallel compensator as

$$\phi(u)=\begin{cases}0, & u<65\\ 0.0025(u-65), & \text{otherwise}\end{cases} \quad (23)$$

Here ϕ_1 is set to be 65 which is smaller a little than the input u showing the zero steady state gain and ϕ_2 is set to be 0.0025 which is greater than minus of the minimum process gain of −0.0023. It satisfies that the steady state gain of Eq. (6) is positive throughout the operating region.

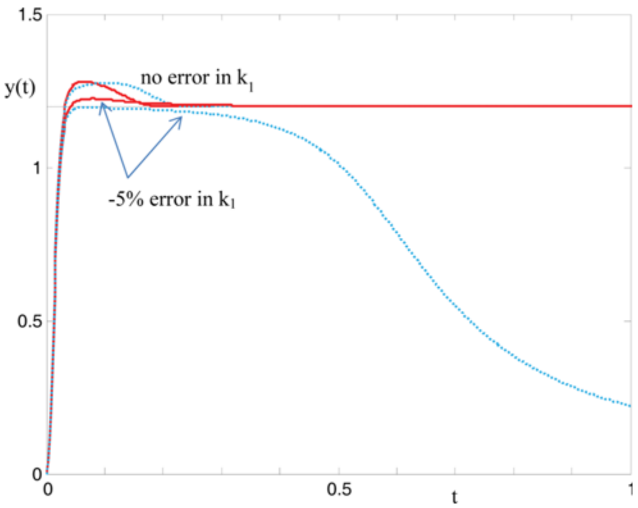

Fig. 8. Responses of the integral controls with (solid line) and without (dotted line) the parallel compensator for example 2 process.

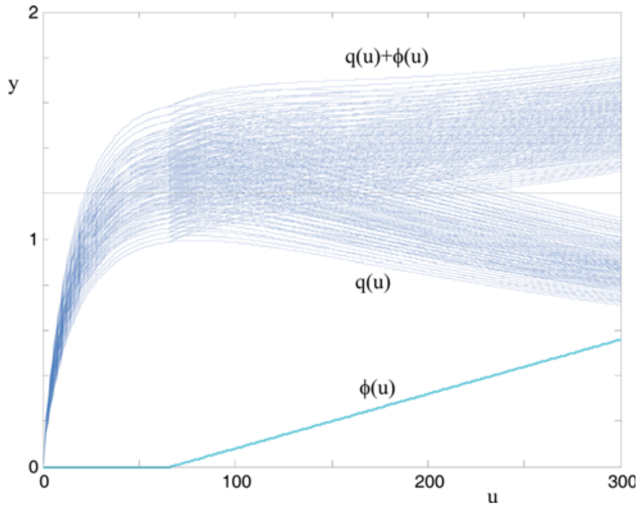

Fig. 9. Functions of q(u), ϕ(u) and q(u)+ϕ(u) for example 2 process whose four parameters have ±10% errors.

Fig. 8 shows the closed-loop performances of integral control systems with and without the parallel compensator (23). The integral gain, k_I, is set to 5,000 and the sampling time is set to 0.002. The set point is r=1.2. Both integral control systems with and without the parallel compensator (23) can steer the process to the given set point of 1.2 from origin. When k_1 has −5% error (k_1=47.5 instead of 50), the proposed integral control system with the parallel compensator of Eq. (23) can steer the process output to the set point. On the other hand, the control system without the parallel compensator fails to steer the process output to the set point.

Fig. 9 shows the steady state input-output relationships with and without the parallel compensator (23) when ±10% errors are introduced in the four process parameters k_1, k_2, k_3 and x_{1f}. We can see that the parallel compensator (23) makes the steady state gain remain positive. The proposed control system with the parallel compensator of Eq. (23) maintains its stability robustly for all such process parameter changes. On the other hand, for process changes where the maximum of y is less than r=1.2, previous model-based control systems [3,5,15,16] may suffer from instability.

CONCLUSION

Input multiplicities are often considered to be what should be removed in the process design step because there have been no simple control systems ensuring robust stability until now. For model-based control systems, process models predicting the sign change of steady state gains accurately are required. Otherwise, complicated schemes estimating process steady state gains adaptively should be equipped. Here, to relieve such difficulties for the process showing input multiplicities, a simple control system is proposed. To cope with sign changes of the process steady state gain in the process with input multiplicity, a parallel compensator is introduced. It is shown theoretically and with simulations that the control difficulties due to the sign change of the process steady state gain can be mitigated. In addition, the parallel compensator can be designed easily based on the process steady state gain information and tuned in the field.

ACKNOWLEDGEMENTS

This work was supported by the NRF grant funded by the Korea government through the Mid-career Research Program (2014004928).

NOTATION

C(s) : proportional and high order part in the feedback controller
c_1, c_2 : positive constants
f(x, z) : a set of n nonlinear functions
h(x) : a nonlinear function
$k_I=\varepsilon\tilde{k}_I$: integral controller gain
k_1, k_2, k_3 : process parameters in the Van de Vusse reactor model
n : number of state variable x
q(u) : nonlinear function relating the process input and output at the steady state
r : set point
t : time
u : input variable
v : output of the integral controller
V(u) : a Lyapunov function
$W(\tilde{x}, u)$: a Lyapunov function
x : n dimensional state vector
$\tilde{x}=x-\xi(z)$: deviation variable for x
x_{1f} : feed concentration in the Van de Vusse reactor model
y : output variable
y_ϕ : compensated output variable

Subscript

ss : steady state value

Greek Letter

ε : a parameter for the singular perturbation
$\xi(u)$: a nonlinear function
$\phi(v)$: a nonlinear function
ϕ_1, ϕ_2 : design parameters for $\phi(v)$
τ : scaled time variable (t/ε)
θ : time delay

REFERENCES

1. L. B. Koppel, *AIChE J.*, **28**, 935 (1982).
2. S. K. Dash and L. B. Koppel, *Chem. Eng. Commun.*, **84**, 129 (1989).
3. P. B. Sistu and B. W. Bequette, *Chem. Eng. Sci.*, **50**, 921 (1995).
4. F. Ahmed, H. J. Cho, J. K. Kim, N. U. Seong and Y. K. Yeo, *Korean J. Chem. Eng.*, **32**, 1029 (2015).
5. H. Ahn, K. S. Lee, M. Kim and J. Lee, *Korean J. Chem. Eng.*, **31**, 6 (2014).
6. A. Uppal, W. H. Ray and A. B. Poore, *Chem. Eng. Sci.*, **29**, 967 (1974).
7. A. Zheng, V. Grassi and Meski, *Ind. Eng. Chem. Res.*, **37**, 1836 (1998).
8. A. Rosales-Quintero and F. D. Vargas-Villamil, *Chem. Eng. Res. Des.*, **89**, 586 (2011).
9. H. Seki, S. Hoshino and Y. Naka, *17th World Congress IFAC*, Seoul, Korea (2008).
10. A. P. Singh and M. Nikolaou, *J. Process Control*, **23**, 294 (2013).
11. I. Malinen and J. Tanskanen, *Comput. Chem. Eng.*, **34**, 1761 (2010).
12. K. Ma, H. Valdes-Gonzalez and I. D. L. Bogle, *J. Process Control*, **20**, 241 (2010).
13. H. Santoso, D. Hioe, J. Bao and P. L. Lee, *J. Process Control*, **22**, 156 (2012).
14. H. Santoso, D. Hioe, J. Bao and P. L. Lee, *J. Process Control*, **19**, 464 (2009).
15. M. Chidambaram and G. P. Reddy, *Computers Chem. Eng.*, **19**, 249 (1995).
16. G. P. Reddy and B. C. Eranna, *J. Artificial Intelligence: Theory and Application*, **1**, 48 (2010).
17. B. Aufderheide and B. W. Bequette, *Comput. Chem. Eng.*, **27**, 1079 (2003).
18. Y. Chikkula, J. H. Lee and B. A. Ogunnaike, *AIChE J.*, **44**, 2658 (1998).
19. H. K. Khalil, *Nonlinear systems*, Prentice-Hall, New Jersey (1996).
20. R. A. Wright, C. Kravaris and N. Kazantzis, *AIChE J.*, **47**, 1805 (2001).

APPENDIX (STABILITY OF A NONLINEAR TWO TIME SCALE SYSTEM)

Consider a singularly perturbed system

$$\varepsilon\dot{x}=f(x, z) \quad \text{(A1)}$$
$$\dot{z}=g(x, z)$$

with f(0, 0)=0 and g(0, 0)=0. Here $x\in D_x\subset R^n$ where D_x is an open connected set that contains x=0 is the fast-varying variable and $z\in D_z\subset R^m$ where D_z is an open connected set that contains z=0 is the slowly-varying variable. Let $x=\xi(z)$ be an isolated root of f(x, z)=0 and introduce a new variable

$$x=x-\xi(z) \quad \text{(A2)}$$

Then the system (A1) becomes

$$\varepsilon\dot{\tilde{x}}=f(\tilde{x}+\xi(z), z)-\varepsilon\frac{\partial\xi(z)}{\partial z}g(\tilde{x}+\xi(z), z) \quad \text{(A3)}$$
$$\dot{z}=g(\tilde{x}+\xi(z), z)$$

It is assumed that the asymptotical stability of the origin of (A3) implies that of (A1).

Let $W(\tilde{x}, z)$ and V(z) be Lyapunov functions for the fast and slow subsystems of (A3)

$$\varepsilon\dot{\tilde{x}}=f(\tilde{x}+\xi(z), z) \quad \text{(A4)}$$
$$\dot{z}=g(\xi(z), z)$$

satisfying

$$\frac{\partial W}{\partial\tilde{x}}f(\tilde{x}+\xi(z), z)\le-\alpha_1\psi_1^2(\tilde{x}) \quad \text{(A5)}$$

$$\frac{\partial V}{\partial z}g(\xi(z),z)\leq-\alpha_2\psi_2^2(z)$$

where $\psi_1(\tilde{x})$ and $\psi_2(z)$ are positive definite functions and α_1 and α_2 are positive constants. These Lyapunov functions are assumed to satisfy additional conditions

$$W_1(\tilde{x})\leq W(\tilde{x},z)\leq W_2(\tilde{x})$$
$$\left[\frac{\partial W}{\partial z}-\frac{\partial W}{\partial \tilde{x}}\frac{\partial \xi}{\partial z}\right]g(\tilde{x}+\xi(z),z)\leq\beta_1\psi_1(\tilde{x})\psi_2(z)+\beta_3\psi_1^2(\tilde{x}) \qquad (A6)$$
$$\frac{\partial V}{\partial z}[g(\tilde{x}+\xi(z),z)-g(\xi(z),z)]\leq\beta_2\psi_1(\tilde{x})\psi_2(z)$$

Here β_1, β_2 and β_3 are nonnegative constants. The condition of Eq. (A6) will be satisfied if for some nonnegative constants γ_1 to γ_5

$$\left\|\frac{\partial W(\tilde{x},z)}{\partial \tilde{x}}\right\|\leq\gamma_1\psi_1(\tilde{x}),\ \left\|\frac{\partial W(\tilde{x},z)}{\partial z}\right\|\leq\gamma_2\psi_1(\tilde{x})$$
$$\|g(\tilde{x}+\xi(z),z)-g(\xi(z),z)\|\leq\gamma_3\psi_1(\tilde{x}) \qquad (A7)$$
$$\left\|\frac{\partial V(z)}{\partial z}\right\|\leq\gamma_4\psi_2(z),\ \|g(\xi(z),z)\|\leq\gamma_5\psi_2(z)$$

Theorem (Khalil [19]): Consider the singularly perturbed system (A1). Assume that there are Lyapunov functions, $W(\tilde{x}, z)$ and $V(z)$, which satisfy (A5) and (A6). Then, the origin of (A1) is asymptotically stable for all $0<\varepsilon<\varepsilon^*=\alpha_1\alpha_2/(\alpha_2\beta_3+\beta_1\beta_2)$.

Proof: This theorem can be proved from the composite Lyapunov function candidate

$$Y(\tilde{x},z)=dW(\tilde{x},z)+(1-d)V(z) \qquad (A8)$$

Details are given in Khalil [19].