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# Third quadrant Nyquist point for the relay feedback autotuning of PI controllers

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**Abstract**−The original relay feedback autotuning method of Astrom and Hagglund [1] is based on the Nyquist point at the phase angle of $-\pi$ (the critical frequency). Recently, Friman and Waller [8] showed that the critical frequency is too high to tune PI controllers and proposed an autotuning method that finds a Nyquist point at the third quadrant through the two-channel relay. Here, the method to find Nyquist points in the third quadrant is revisited and adaptive relay feedback method which can be applied to noisy processes is proposed. It is shown that the bandwidths of PI control systems and the first-order plus time delay model identifications support the Nyquist point at the third quadrant. Nyquist points at the third quadrant can be obtained by introducing a filter and hysteresis to the relay feedback loop. However, the filter time constant and the size of hysteresis should be adjusted iteratively because their phase shifts are dependent on the resulting relay oscillation frequency. Simulations show that this adaptive relay feedback method finds a given Nyquist point at the third quadrant accurately under noisy environments and provides excellent PI control systems.

Key words: Nyquist Points, Third Quadrant, PI Controller Tuning, Filters, Hysteresis

## INTRODUCTION

Since Astrom and Hagglund [1] introduced the relay feedback autotuning method for PID controllers, many variations have been proposed for better autotuning of PID controllers [2-4]. These include a saturation relay [4], relay with a P control preload [5] and a two level relay [6]. To obtain a Nyquist point other than the critical point, a relay with hysteresis or a dynamic element such as time delay has been used [2,7]. A two-channel relay has been proposed to obtain a Nyquist point information corresponding to a given phase angle [8,9]. Methods to reject unknown load disturbances and restore symmetric relay oscillations have been available [10,11]. Recently, Lee et al. [12] used integrals of responses to improve the accuracy of ultimate data identification by reducing high order harmonics.

The critical frequency is too high in general to use the frequency response information at that frequency for the PI controller tuning. However, there are few methods to tune controllers with frequency response information below the critical point. Recently, Friman and Waller [8] proposed methods to find a Nyquist point in the third quadrant and to tune PI/PID controllers. They used two-channel relay to identify the frequency response information at the third quadrant Nyquist point without iteration and proposed tuning rules for PI/PID controllers. In this research, we show that the closed-loop bandwidths of PI control systems and the FOPTD models also support the usage of the frequency response information at the third quadrant Nyquist point for the PI controller tuning.

For noisy environments, relay feedback methods can suffer from chattering. This chattering does not affect characteristics of relay oscillations such as stability. However, for some processes, chattering is not allowed. To avoid this chattering, a relay with hysteresis can be used. However, its switching period fluctuates and the size of hysteresis should be adjusted iteratively for a given phase angle [2]. A low-pass filter in the feedback loop may be used to find a frequency response information at the third quadrant Nyquist point, but the filter time constant should be adjusted iteratively to obtain the process information at a given phase angle. Changing the filter time constant requires a transient period and causes the convergence problem. To find process information at the given third quadrant Nyquist point for noisy processes, filter methods free from the transient period are proposed.

## USAGE OF NYQUIST POINTS IN THE THIRD QUADRANT

### 1. Gain Margin and Phase Margin of PI Control System

For the PI controller,

$$C(s) = k_c\left(1 + \frac{1}{\tau_I s}\right) \tag{1}$$

the phase angle of $\angle C(j\omega) = -\arctan(1/\tau_I\omega)$ is always negative for the positive integral time $\tau_I$ and the angular frequency $\omega$. Hence the frequency response of the process $G_p(s)$ in the area **A** in Fig. 1 is mapped to the frequency response for $G_p(s)C(s)$ in the area **B**. Because the gain margin and the phase margin of feedback system with a process $G_p(s)$ and a controller C(s) can be determined from the frequency responses in the area **B**, the frequency responses of area **B** for $G_p(s)C(s)$ and consequently those of area **A** for $G_p(s)$ are important to design PI control systems. This figure shows that the ultimate frequency is rather high for PI controller tuning. Based on this interpretation of PI control systems, Friman and Waller [8] proposed a two-channel relay feedback method to identify the frequency response in area A and tuning rules for PI controllers using this Nyquist point in the third quadrant.

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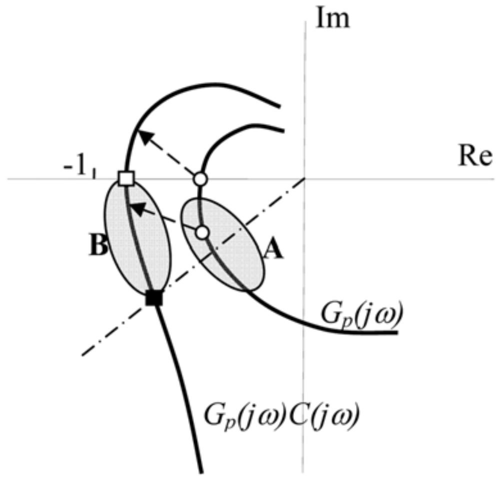


**Fig. 1. Nyquist plots for a typical process, $G_p(j\omega)$, and a PI control system, $G_p(j\omega)C(j\omega)$, (Shaded regions are important frequency regions to guarantee the gain and phase margins of PI control systems).**

## 2. Bandwidths of PI Control Systems

Models for the control system design affect the closed loop responses. One of the criteria for the optimal models can be [13]

$$J=\left\|\frac{G_p(s)C(s)}{1+G_p(s)C(s)}-\frac{G_m(s)C(s)}{1+G_m(s)C(s)}\right\|_2 \tag{2}$$

where $G_m(s)$ is an approximate model of the process $G_p(s)$ and the signal norm $\|f(s)\|_2$ is such that $\|f(s)\|_2\equiv\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}f(-j\omega)f(j\omega)d\omega}$. Since the controller C(s) is designed based on the model $G_m(s)$, the problem to minimize the cost J of Eq. (2) usually needs iterations [13]. Under the assumption that $G_p(s)\approx G_m(s)$, we have

$$\begin{aligned}J&\approx\left\|\frac{G_p(s)C(s)}{1+G_p(s)C(s)}-\frac{G_m(s)C(s)}{1+G_p(s)C(s)}\right\|\\&=\left\|\frac{G_p(s)C(s)}{1+G_p(s)C(s)}\frac{G_p(s)-G_m(s)}{G_p(s)}\right\|_2\end{aligned} \tag{3}$$

This shows that the modeling errors below the bandwidth frequency of the closed loop system, $G_p(s)C(s)/(1+G_p(s)C(s))$, should be small [14]. In other words, the frequency responses below the bandwidth of the closed loop system are more important than those over the bandwidth.

Here the phase angle of $G_p(s)$ at the bandwidth of the PI control system is investigated. For the first order plus time delay process

$$G_p(s)=\frac{ke^{-\theta s}}{\tau s+1} \tag{4}$$

the internal model control (IMC) tuning rule is intended for the closed loop time constant to be $\lambda$ and for PI controller the recommended $\lambda$ is $1.7\theta$. This means that the closed loop bandwidth is approximately $\omega_{WB}=1/\lambda=1/1.7\theta$. At this bandwidth, we have

$$\angle G_p(s)_{s=j\frac{1}{1.7\theta}}=-\frac{1}{1.7}-\arctan\left(\frac{\tau}{1.7\theta}\right) \tag{5}$$

The angle of Eq. (5) is between $-124^{\circ}$ and $-34^{\circ}$ as the ratio $\theta/\tau$ between 0 and infinity.

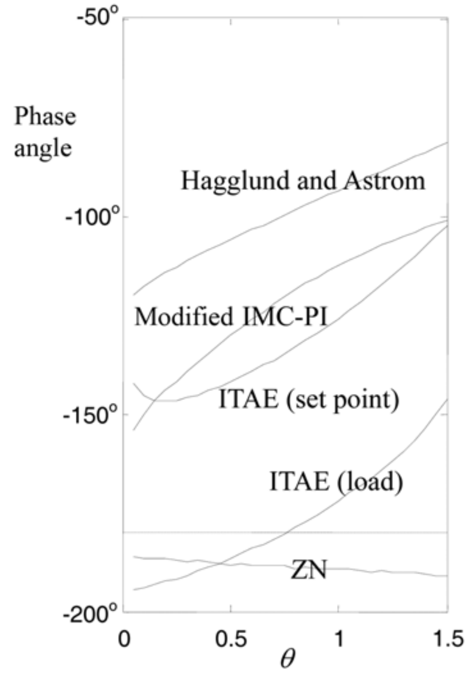


**Fig. 2. Phase angles of process at the bandwidth frequency of PI control systems for the first order plus time delay process, $G_p(s)=\exp(-\theta s)/(s+1)$. (modified IMC-PI [16]: $k_c=0.5/\theta$, $\tau_I=\min(1, 8+\theta)$).**

The phase angles at the bandwidths of PI control systems for several tuning rules are shown in Fig. 2. Tuning rules of the ZN (Ziegler-Nichols) and the load ITAE (integral of time absolute error) that are known to be very aggressive [15] show phase angles over $-180^{\circ}$. On the other hand, tuning rules of the simplified IMC [16] and the set point ITAE that are conservative [15] show phase angles below $-180^{\circ}$. Considering that conservative tuning rules are preferred in the field, frequency responses below the critical point are more important than those near the critical point.

## 3. FOPTD Models

There are many tuning rules based on the FOPTD (first order plus time delay) model. This FOPTD model can be obtained from the step test data. The FOPTD model can also be obtained from the steady state information $G_p(0)$ and a frequency response information $G_p(j\omega)$ at a certain frequency $\omega$ as

$$\begin{aligned}G_m(s)&=\frac{k_m e^{-\theta_m s}}{\tau_m s+1}\\k_m&=G_p(0)\\\tau_m&=\frac{1}{\omega}\sqrt{\frac{k_m^2}{|G_p(j\omega)|^2}-1}\\\theta_m&=\frac{-\angle G_p(j\omega)-\arctan(\tau_m\omega)}{\omega}\end{aligned} \tag{6}$$

Here we consider the problem to find the optimal frequency $\omega$ that minimizes the cost

$$J_2=\int_0^{\infty}(y(t)-y_m(t))^2dt \tag{7}$$

where y(t) is the process step response and $y_m(t)$ is the step response

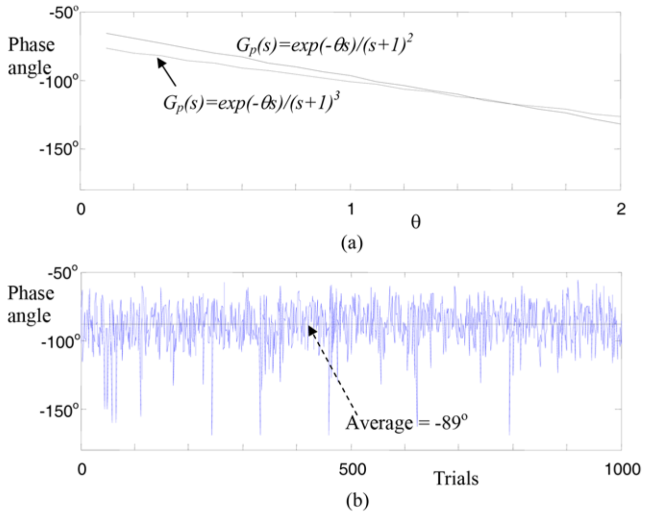


**Fig. 3. Phase angle of process at the optimal frequency for FOPTD models. (a) Second and third order plus time delay process, (b) random processes of $G_p(s)=(ds+1)\exp(-es)/((s+1)(as+1)(bs+1)(cs+1))$.**

of its FOPTD model of Eq. (6). For this, for a given frequency $\omega$, we calculate the FOPTD model $G_m(s)$ of Eq. (6) and the cost $J_2$ of Eq. (7). With changing the frequency, we find the optimal frequency $\omega$ numerically. Fig. 3(a) shows the phase angles at the optimal frequencies for the second and third order plus time delay models and Fig. 3(b) shows the phase angles for the 1000 random processes of $G_p(s)=(ds+1)\exp(-es)/((s+1)(as+1)(bs+1)(cs+1))$ where a, b, c, d, and e are random numbers between 0 and 1. The optimal frequencies providing the optimal FOPTD models are far below than the critical frequency of the phase angle at $-180^{\circ}$.

## RELAY WITH ADAPTIVE FILTER AND HYSTERESIS

Relay in the feedback loop can make the closed system oscillate at the approximate critical frequency of process. From this oscillation, the Nyquist point on the negative real axis can be obtained. For the oscillation below the critical frequency, a low pass filter can be used. The low pass filter can also attenuate the noise. However, to obtain the Nyquist point at a given phase angle, the filter time constant should be adjusted iteratively. A filter with changing time constants needs a transient period and its implementation is not easy. The hysteresis relay can also change the oscillation period. By combining the hysteresis relay and the low pass filter, an iterative method to obtain the Nyquist point at a given phase angle is proposed.

Consider a system shown in Fig. 4. The filter is

$$F(s)=\frac{\alpha s+1}{(fs+1)^2} \tag{8}$$

where f is a given filter time constant and $\alpha$ is a adjustable parameter. The relay has a hysteresis of $\eta a$ as in Fig. 4. The phase shift due to both filter and hysteresis is

$$\phi=\arctan(\alpha\omega)-2\arctan(f\omega)+\arctan\left(\frac{\eta}{\sqrt{1-\eta^2}}\right) \tag{9}$$

For a given filter time constant f, the phase angle of Eq. (9) can be

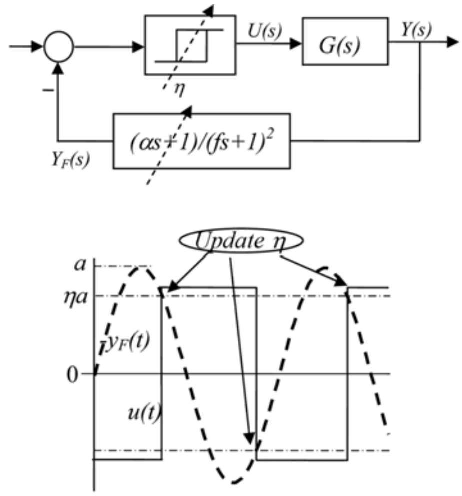


**Fig. 4. Relay feedback system with adaptive filter and hysteresis.**

what we want by adjusting $\alpha$ or $\eta$. Because the oscillation frequency $\omega$ is varying as $\alpha$ or $\eta$, it should be changed iteratively.

Implementation steps are as follows:

**Step 1:** Choose the phase angle $\phi$ and filter time constant f. Set $\alpha$=f and $\eta$=0.

**Step 2:** Apply the relay feedback.

**Step 3:** Obtain the oscillation period p and calculate $\omega=2\pi/p$. Calculate $\eta$ by solving Eq. (9) as

$$\eta=\sin(\phi-\arctan(\alpha\omega)+2\arctan(f\omega)) \tag{10}$$

If $|\eta|<0.5$, use this. Otherwise, calculate a by solving Eq. (9) as

$$\alpha=\frac{\tan\left(\phi+2\arctan(f\omega)-\arctan\left(\frac{\eta}{1-\eta^2}\right)\right)}{\omega} \tag{11}$$

and use this. Update $\eta$ (Eq. (10)) or $\alpha$ (Eq. (11)) between relay switch-

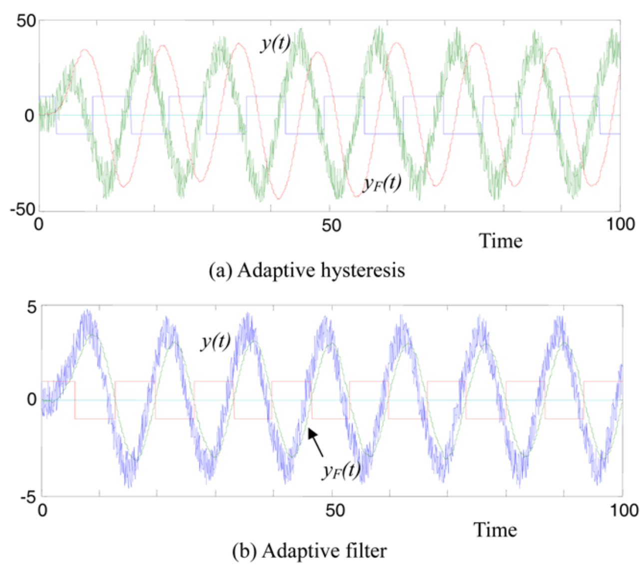


**Fig. 5. Responses of the adaptive filter/hysteresis relay feedback system.**

ings (near the maximum and minimum of the filter output).

**Step 4:** Repeat Step 3 for 4 or more periods.

We, if required, can increase the number of iterations in **Step 4**. Fig. 5 shows responses of this relay feedback system under noisy environment.

## APPLICATIONS

The proposed method is applied to second order plus time delay process,

$$G_p(s)=\frac{\exp(-\theta s)}{(s+1)^2} \tag{12}$$

Filter time constant used is f=0.5. Oscillation frequencies, $\omega=2\pi/p$, are obtained as in Table 1 and compared with the conventional relay feedback method [1]. We can see that oscillation frequencies obtained by the proposed method are as accurate as those by the conventional relay feedback method. Increased errors for small time delays are due to the sensitivities of critical frequencies. The critical frequencies are very sensitive for the process (12) with very small time delays.

**Table 1. Relay feedback oscillation frequencies for the second order plus time delay process, $G_p(s)=(\exp(-\theta s))/(s+1)^2$ (the proposed two-filter method for the phase lag of −150° and the conventional relay feedback method for the phase lag of −180°)**

| Time delay ($\theta$) | Sampling time | Phase lag | Exact frequency | Relay feedback oscillation frequency (Percent error) |
|---|---|---|---|---|
| 0.01 | 0.001 | −150° | 3.488 | 3.353 (−3.9%) |
| | | −180° | 14.14 | 12.15 (−14%) |
| 0.1 | 0.01 | −150° | 2.467 | 2.398 (−2.8%) |
| | | −180° | 4.440 | 3.927 (−12%) |
| 0.5 | 0.01 | −150° | 1.414 | 1.409 (−0.4%) |
| | | −180° | 1.930 | 1.870 (−3.1%) |
| 1.0 | 0.02 | −150° | 1.024 | 1.033 (0.9%) |
| | | −180° | 1.310 | 1.298 (−0.9%) |

For various processes including over-damped processes with time delays, inverse response process, and integral process, tuning results by the proposed method and the conventional relay feedback method are compared. For the conventional relay feedback method, the critical frequency points (phase lag of −180°) are obtained, the Ziegler-Nichols tuning rule [2] is applied. For the proposed method, third quadrant Nyquist points (phase lag of −150°) are obtained and the tuning rule by Friman and Waller [8] is used, whereas the proportional gain is $k_c=r_s\cos(\varphi_s-\varphi_p)/|G_p(j\omega_{150^o})|$ and the integral time constant is, $\tau_I=1/\omega_{150^o}\tan(\varphi_p-\varphi_s)$, where $r_s=0.5$, $\varphi_s=\pi/12$, $\varphi_p=\pi/6$, and $\omega_{150^o}$ is the angular frequency at the phase angle of −150°. Table 2 shows relay oscillation data and tuning results. Fig. 6 shows tuning performance. The conventional relay feedback method based on the Ziegler-Nichols tuning rule shows unstable closed-loop response for $\theta$=0.01 and oscillatory closed-loop response for $\theta$=0.1 for the second order plus time delay process of Eq. (12). On the other hand, the proposed method with a third quadrant Nyquist point and a tuning rule by Friman and Waller [8] shows excellent performance throughout $\theta$. For other processes, the proposed method shows similar or better performance.

When the process steady state gain in addition to a process Nyquist point information in the third quadrant is available, a first order plus time delay (FOPTD) model can be obtained [4] and model-based tuning rules such as the IMC-PI tuning [15] can be applied. IMC-PI tuning performances are compared for FOPTD models obtained from the proposed Nyquist point in the third quadrant and the conventional critical point. Table 3 shows FOPTD models obtained and tuning results. Tuning performances are very similar except for small time delay processes as shown in Fig. 7.

**Table 2. Relay feedback identifications and PI controller tuning for various processes**

| Process (G(s)) | Relay method | Relay oscillation | | PI controller tuning | | IAE* |
|---|---|---|---|---|---|---|
| | | Amplitude | Period | $k_c$ | $\tau_I$ | |
| $\frac{e^{-0.01s}}{(s+1)^2}$ | Proposed | 0.184 | 1.892 | 3.309 | 0.985 | 2.931 |
| | Conventional | 0.007 | 0.484 | 79.169 | 0.403 | Inf |
| $\frac{e^{-0.1s}}{(s+1)^2}$ | Proposed | 0.245 | 2.614 | 2.488 | 1.361 | 2.973 |
| | Conventional | 0.053 | 1.490 | 10.988 | 1.242 | 22.289 |
| $\frac{e^{-0.5s}}{(s+1)^2}$ | Proposed | 0.455 | 4.460 | 1.337 | 2.322 | 4.140 |
| | Conventional | 0.227 | 3.320 | 2.546 | 2.767 | 6.624 |
| $\frac{e^{-s}}{(s+1)^2}$ | Proposed | 0.663 | 6.100 | 0.917 | 3.175 | 6.912 |
| | Conventional | 0.392 | 4.780 | 1.476 | 3.983 | 7.435 |
| $\frac{(-2s+1)e^{-s}}{(s+1)^2}$ | Proposed | 0.924 | 7.460 | 0.435 | 1.520 | 8.610 |
| | Conventional | 0.602 | 5.920 | 0.965 | 4.933 | 10.361 |
| $\frac{e^{-s}}{(s+1)^3}$ | Proposed | 0.700 | 8.620 | 0.870 | 4.487 | 10.301 |
| | Conventional | 0.414 | 6.840 | 1.399 | 5.700 | 10.941 |
| $\frac{1}{(s+1)^6}$ | Proposed | 0.717 | 13.480 | 0.849 | 7.017 | 16.517 |
| | Conventional | 0.428 | 10.900 | 1.352 | 9.083 | 17.693 |
| $\frac{e^{-s}}{s(s+1)}$ | Proposed | 2.409 | 12.400 | 0.253 | 6.455 | 18.042 |
| | Conventional | 0.950 | 7.520 | 0.609 | 6.267 | 22.378 |

*IAE: integral of absolute errors

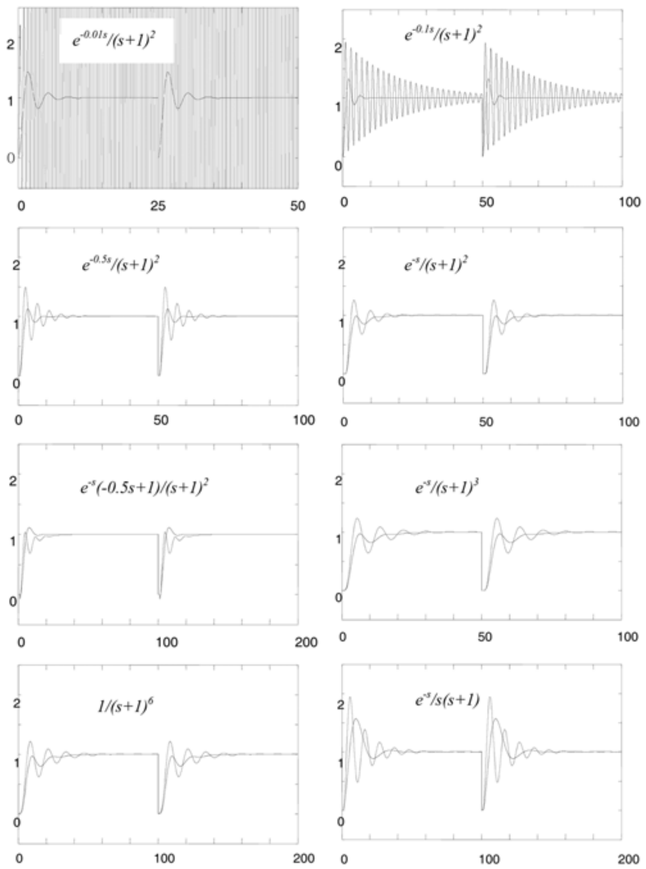


**Fig. 6. Closed-loop responses of PI control systems tuned by the proposed method (solid line) and the conventional relay feedback method (dashed line) for various processes.**

## CONCLUSION

Nyquist points in the third quadrant are shown to be better to tune PI controllers. Relay feedback methods to identify third quadrant Nyquist points which can be applied to noisy processes and to tune PI controllers are proposed.

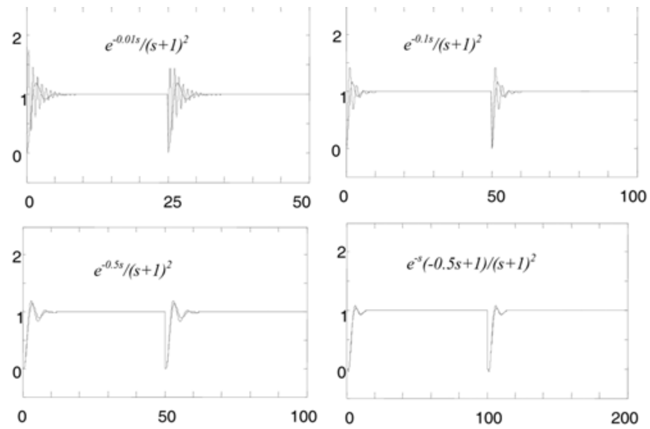


**Fig. 7. Closed-loop responses of PI control systems tuned by the IMC-PI tuning method for FOPTD models obtained by the proposed method (solid line) and the conventional relay feedback method (dashed line) for various processes.**

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**Table 3. Relay feedback identification of FOPTD models and IMC-PI tuning for various processes**

| Process (G(s)) | Relay method | Identification parameters | | PI controller tuning | | IAE* |
|---|---|---|---|---|---|---|
| | | $\tau$ | $\theta$ | $k_c$ | $\tau_I$ | |
| $\frac{e^{-0.01s}}{(s+1)^2}$ | Proposed | 2.062 | 0.359 | 3.673 | 2.242 | 1.936 |
| | Conventional | 13.417 | 0.121 | 65.280 | 13.477 | 2.215 |
| $\frac{e^{-0.1s}}{(s+1)^2}$ | Proposed | 2.029 | 0.521 | 2.584 | 2.289 | 2.424 |
| | Conventional | 4.471 | 0.388 | 7.077 | 4.665 | 2.764 |
| $\frac{e^{-0.5s}}{(s+1)^2}$ | Proposed | 1.854 | 1.003 | 1.381 | 2.355 | 4.149 |
| | Conventional | 2.263 | 0.951 | 1.694 | 2.739 | 4.304 |
| $\frac{e^{-s}}{(s+1)^2}$ | Proposed | 1.590 | 1.549 | 0.898 | 2.364 | 5.982 |
| | Conventional | 1.785 | 1.501 | 0.994 | 2.536 | 6.074 |
| $\frac{(-0.5s+1)e^{-s}}{(s+1)^2}$ | Proposed | 1.127 | 2.207 | 0.594 | 2.230 | 7.749 |
| | Conventional | 1.250 | 2.089 | 0.646 | 2.294 | 7.725 |
| $\frac{e^{-s}}{(s+1)^3}$ | Proposed | 2.085 | 2.235 | 0.843 | 3.203 | 8.537 |
| | Conventional | 2.396 | 2.174 | 0.942 | 3.483 | 8.722 |
| $\frac{1}{(s+1)^6}$ | Proposed | 3.148 | 3.530 | 0.819 | 4.913 | 13.505 |
| | Conventional | 3.662 | 3.492 | 0.911 | 5.409 | 13.827 |

*IAE: integral of absolute errors

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