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Finite-Amplitude Surface Waves on a Thin Film Flow Subject to a Unipolar-Charge Injection

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Abstract−The interaction of an electric field with a dielectric liquid film is investigated as it drains under gravity down an inclined plane electrode emitting uniform positive ions into the liquid region. By applying long-wave approximation to the governing equations, the evolution equation for the free surface is derived up to the first order of a thin film parameter ξ. To investigate the space charge effect on the development of a finite-amplitude surface wave, a neutral stability condition is obtained as a critical Reynolds number through a linear stability analysis, and the amplitude and velocity of a periodic disturbance are also calculated within a supercritically stable flow region. The presence of a unipolar space charge in the fluid makes a steady surface wave take on even higher amplitude and faster wave speed compared with the case of no space charge.

Key words: Finite-Amplitude Surface Waves, Unipolar-Charge Injection, Dielectric Liquid Film, Critical Reynolds Number

INTRODUCTION

The hydrodynamic study under an effect of electric force has recently attracted much attention and also extensively studied for many new processes such as ion-drag pump, turbulent mixer, and heat convecting system, etc. by using a dielectric liquid containing space charges as a working fluid. The transfer rates of momentum and heat are known to be much increased owing to the unipolar charges injected into the liquid medium [Atten, 1996].

Here, as a new basic problem dealing with the effects of free charges on the fluid motion, the surface wave behavior of a dielectric liquid layer will be considered when it is drained by gravity down an inclined plane electrode and at the same time it is subjected to a unipolar injection from this bottom electrode with some electric potential. Another plane electrode with zero potential is mounted above the liquid film at some distance parallel to the bottom electrode. Air is occupied between the liquid layer surface and the top electrode.

In somewhat different applications from the present, Castellanos et al. [1992] studied the behavior of the unipolar injection induced instabilities in plane Poiseuille and Couette flows where they determined the resulting convective patterns and the instability thresholds to get the insight of the physical mechanisms. In the absence of the space charge effect, and as far as the nonlinear electrohydrodynamic film flows have been concerned, Kim et al. [1992], González et al. [1996], and Kim [1997, 2003] have examined and analyzed many aspects of a thin liquid layer flowing down an inclined plane under an applied electric field. Kim et al. [1992] systematically scrutinized the effect of an electrostatic force on the film flows by driving analytical equations of motion in the limits of small and large Reynolds numbers. González et al. [1996] observed the stability of the nonlinear electrohydrodynamic waves based on the nonlinear evolution equation for the film height using the long-wave approximation. In order to focus on the linear and nonlinear stabilities, they considered the fluid was a perfect electric conductor and the electric field was uniformly applied from infinity. Kim [1997] examined the nonlinear surface-wave instabilities numerically with a Fourier-spectral method, and he also confirmed the existence of pulse-like solitary waves with the help of a global bifurcation theory and demonstrated their developing processes numerically in his work [2003].

The main purpose of present study is to derive the equations and proper boundary conditions governing the electrohydrodynamics of a thin film flow into which unipolar-charged ions are injecting. Utilizing a long-wave approximation to the flow system makes it possible to decouple the dominant terms of electric potentials and space charge from the fluid dynamics, which pushes the system to be exposed to analytical solutions. After obtaining a surface-wave evolution equation of the flow region, as in other general hydrodynamic problems, the electrohydrodynamic instability is carried out on a linearly perturbed surface wave to determine the onset conditions at which the layer will be susceptible to the development to turbulence.

MATHEMATICAL FORMULATION

Assuming the liquid is isothermally incompressible, viscous and dielectric, the thin layer runs down an inclined plane under the action of gravity g. The plane supporting the fluid layer acts as an injector with an electric potential V and a positive-ion charge density q_i. The space charges will be spreaded out through the liquid phase and exert the subsequent Coulomb force on the flow. It is assumed that the migration of the charged particles relative to the fluid motion is insignificant because of their relatively small relaxation time (defined by the ratio of electric permittivity to conductivity of the liquid) and thus the charge is frozen to the liquid, and the surface charge density on the free surface is also assumed inconsequential.

In order to solve the governing field equations with proper boundary conditions, it is convenient to take their dimensionless forms. Only two-dimensional case will be considered. Hence, the bottom

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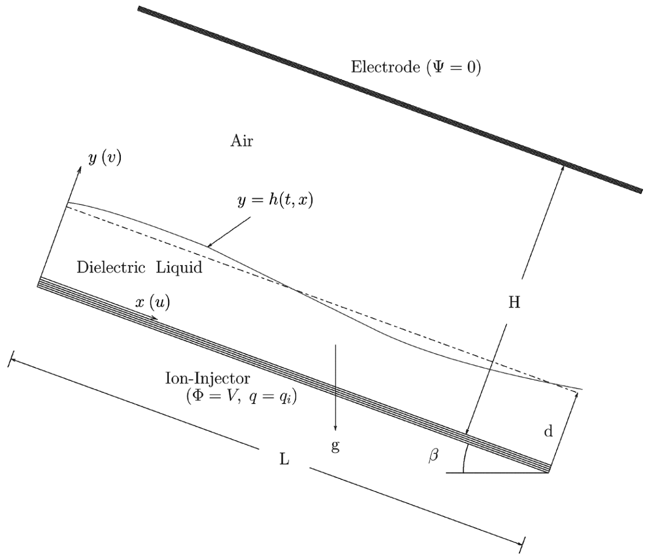


Fig. 1. The physical configuration of the plane flow subjected to a unipolar-ion injection.

plane electrode is assumed to make an angle β with the horizontal and the coordinate system is taken such that the x axis is parallel to the plane while the y axis is perpendicular to it as shown in Fig. 1. Above the liquid layer there is an air which is assumed free of charge, and the density and viscosity of air will be taken no account of here in comparison with those of liquid. Within the air region at a distance H from the bottom plane of length L locates an uncharged electrode with the same dimension of the bottom one. Suppose that d is defined as the characteristic thickness of the primary film flow and L is the length scale of the disturbance in the x direction, then the parameter ξ=d/L will appear in the dimensionless equations. Letting V be the unit of electric potential, q_i the unit of space charge density, d the unit of length in the y direction, U_0 the unit of velocity for u in the x direction (the specific U_0 will be chosen later), ξU_0 the unit of velocity for v in the y direction, L/U_0 the unit of time t, and ρU_0^2 the unit of pressure p, the dimensionless field equations and boundary conditions can be determined.

First, for the electrical field in the liquid regime, the Gauss law and conservation of space charge due to the convection of charge with the fluid motion and the conduction current [Atten, 1996] yield, respectively, the following equations:

$$\xi^2\Phi_{xx}+\Phi_{yy}=-Cq \tag{1}$$

and

$$\xi q_t+\xi(uq_x+vq_y)+R[(\xi^2 q_x\Phi_x+q_y\Phi_y)+q(\xi^2\Phi_{xx}+\Phi_{yy})]=0, \tag{2}$$

where the dimensionless numbers C and R are defined by

$$C=\frac{q_i d^2}{\varepsilon_f V} \tag{3}$$

and

$$R=\frac{K_m V}{U_0 d}. \tag{4}$$

Φ and q are the electrical potential and space charge density in the liquid, respectively, and the subscripts stand for the partial derivatives. ε_f and K_m denote the electric permittivity and ion mobility in the liquid. C measures the relative importance of space charge to the applied electric potential and R is similar to the Prandtl number in the hydrodynamics.

For the fluid dynamics, the continuity equation becomes

$$u_x+v_y=0, \tag{5}$$

while the x and y components of the momentum are, respectively,

$$\xi(u_t+uu_x+vu_y)=-\xi p_x+\frac{1}{Re}(\xi^2 u_{xx}+u_{yy})-\xi qNe\Phi_x+\frac{\sin\beta}{Fr^2} \tag{6}$$

and

$$\xi^2(v_t+uv_x+vv_y)=-p_y+\frac{\xi}{Re}(\xi^2 v_{xx}+v_{yy})-qNe\Phi_y-\frac{\cos\beta}{Fr^2}, \tag{7}$$

where the Reynolds number Re, the Froude number Fr and the electrical Newton number Ne have the following definitions:

$$Re=\frac{\rho U_0 d}{\mu}, \tag{8}$$

$$Fr=\frac{U_0}{\sqrt{gd}}, \tag{9}$$

and

$$Ne=\frac{q_i V}{\rho U_0^2}. \tag{10}$$

Here the density and viscosity of the liquid are noted as ρ and μ, respectively.

In the air, there is only one field equation for the electric potential Ψ, i.e.,

$$\xi^2\Psi_{xx}+\Psi_{yy}=0. \tag{11}$$

The dimensionless boundary conditions are still to be specified. Along the injector, y=0, the electric potential and space charge conditions, and the no-slip condition are, respectively,

$$\Phi=1,\ q=1 \tag{12}$$

and

$$u=v=0. \tag{13}$$

On the fluid interface, y=h(t, x), there are two electrical boundary conditions, i.e., the continuities of potential and normal displacement field like

$$\Phi=\Psi \tag{14}$$

and

$$-\xi^2 h_x\Psi_x+\Psi_y=\varepsilon(-\xi^2 h_x\Phi_x+\Phi_y), \tag{15}$$

while for the fluid dynamics there are jump mass, normal and tangential stress conditions, respectively:

$$h_t+uh_x=v, \tag{16}$$

$$\frac{\xi^2}{Ca}\frac{h_{xx}}{(1+\xi^2 h_x^2)^{3/2}}=\frac{Re}{2}(p_a-p)$$
$$+\frac{K}{1+\xi^2 h_x^2}\left(\frac{1}{\varepsilon}-1\right)\{(\xi^2 h_x\Psi_x-\Psi_y)^2+\xi^2\varepsilon(\Psi_x+h_x\Psi_y)^2\}$$

$$+\xi\{\xi^2 h_x^2 u_x - h_x(u_y+\xi^2 v_x)+v_y\}(1+\xi^2 h_x^2)^{-1}, \tag{17}$$

and

$$(1-\xi^2 h_x^2)(u_y+\xi^2 v_x)+2\xi^2 h_x(v_y-u_x)=0. \tag{18}$$

Here p_a is the air pressure, ε is the dielectric constant of liquid defined by $\varepsilon_l/\varepsilon_0$ (ε_0 is the permittivity of free space), and the capillary number Ca and the dimensionless constant K are introduced such as

$$\mathrm{Ca}=\frac{2\mu U_0}{\sigma} \tag{19}$$

and

$$K=\frac{\varepsilon_0 V^2 d}{4\mu U_0}, \tag{20}$$

where σ is denoted as the surface tension of the liquid, and in (20) the MKS electrostatic units are employed.

The last dimensionless boundary condition along y=H is

$$\Psi=0. \tag{21}$$

SOLUTIONS AT LEADING ORDER IN ξ

The thin film limit by taking $\xi \ll 1$ has been considered to get appropriate solutions from the nondimensionalized Eqs. (1)-(21). Assuming C, R, Re, Fr and Ne have order-one values in ξ except Ca which is assumed order of ξ^2 to include the stabilizing effects of surface tension, the solution describing the interface behavior upto order ξ could be sought through a regular perturbation method by expanding the dependent variables in ξ like $\Phi=\Phi_0+\xi\Phi_1+\cdots$, $q=q_0+\xi q_1+\cdots$, $\Psi=\Psi_0+\xi\Psi_1+\cdots$, $u=u_0+\xi u_1+\cdots$, $v=v_0+\xi v_1+\cdots$, and $p=p_0+\xi p_1+\cdots$.

From (1), (2) and (11), the leading-order solutions in ξ for the electric potentials and charge density are obtained as

$$\Phi_0=-\frac{2}{3}c_1(y+c_2)^{3/2}+c_3, \quad \text{for } 0<y<h,$$
$$\Psi_0=c_4 y+c_5, \quad \text{for } h<y<H,$$
$$q_0=\left(\frac{c_2}{y+c_2}\right)^{1/2}, \quad \text{for } 0<y<h, \tag{22}$$

where $c_n(t, x)$ for n=1, ⋯, 5 can be found by solving the following equations which are given from the boundary conditions,

$$c_1=2Cc_2^{1/2},$$
$$c_3=1+\frac{2}{3}c_1 c_2^{3/2},$$
$$c_4=-c_1\varepsilon(h+c_2)^{1/2},$$
$$c_5=-c_4 H,$$
$$c_4 h=c_3-\frac{2}{3}c_1(h+c_2)^{3/2}-c_5. \tag{23}$$

Here it is known that c_1, c_2, c_3 and c_5 are positive whereas c_4 is negative.

By choosing the characteristic unit of velocity as the average velocity of the steady parallel flow, i.e., $U_0=\rho g d^2\sin(\beta)/3\mu$, the leading-order velocity components and pressure are obtained as

$$u_0=3h^2\left[\frac{y}{h}-\frac{1}{2}\left(\frac{y}{h}\right)^2\right], \tag{24}$$

$$v_0=-\frac{3}{2}h_x y^2, \tag{25}$$

$$p_0=p_a+\frac{\cos\beta}{Fr^2}(h-y)-\frac{1}{2}\frac{Ne}{C}\{(\Phi_{0y})^2|_{y=h}-(\Phi_{0y})^2\}$$
$$+\frac{2K}{Re}\left(\frac{1}{\varepsilon}-1\right)(\Psi_{0y})^2-\frac{2}{ReCa}\xi^2 h_{xx}. \tag{26}$$

From these leading-order solutions we have to note that the solutions for the electric fields are independent of the hydrodynamic ones because the solutions are produced from a quasi-steady state in the thin film limit.

FILM EVOLUTION EQUATION

A nonlinear evolution equation describing the film deformation has to be derived to determine the stability of liquid layer subjected to the space charge. In order to include the effects of gravity, viscosity, surface tension, electric field and space charge on the film flow, it is still necessary to find the velocity components one step further to the first order in ξ. However, as long as the parameters stated in the previous section are keeping order unity, the sought solutions for Φ, Ψ and q at the leading order are still enough for the accounts of the electric force effects on the film flow that the procedures to get their next-order solutions of these are not necessary.

By plugging the leading-order solutions (22)-(26) into (5) and (6), the velocity components next to the leading order, i.e., u_1 and v_1 can be gained with the conditions (13) and (18). The results are

$$u_1=3Rehh_x\left(\frac{1}{8}y^4-\frac{1}{2}hy^3+h^3y\right)+\left(3\cot(\beta)-\frac{Ne}{2C}Rec_1^2\right)h_x\left(\frac{y^2}{2}-hy\right)$$
$$-\frac{ReNe}{2\ C}c_1c_{1x}\left(hy^2-\frac{y^3}{3}-h^2y\right)$$
$$-\left\{\frac{\xi^2}{Ca}h_{xxx}-2K\left(\frac{1}{\varepsilon}-1\right)c_4c_{4x}\right\}(y^2-2hy)$$
$$-\frac{ReNe}{18\ C}c_1c_{1x}\{(y+c_2)^3-c_2^3-3(h+c_2)^2y\}$$
$$-\frac{ReNe}{4\ C}c_1^2c_{2x}(y^2-2hy)$$
$$+Re\frac{Ne}{C}c_1c_{3x}\left\{\frac{2}{3}(y+c_2)^{3/2}-\frac{2}{3}c_2^{3/2}-(h+c_2)^{1/2}y\right\} \tag{27}$$

and

$$v_1=-\left[3Rehh_x\left(\frac{1}{40}y^5-\frac{1}{8}hy^4+\frac{1}{2}h^3y^2\right)\right.$$
$$+\left(3\cot(\beta)-\frac{Ne}{2C}Rec_1^2\right)h_x\left(\frac{y^3}{6}-\frac{1}{2}hy^2\right)$$
$$-\frac{ReNe}{2\ C}c_1c_{1x}\left(\frac{1}{3}hy^3-\frac{y^4}{12}-\frac{1}{2}h^2y^2\right)$$
$$-\left\{\frac{\xi^2}{Ca}h_{xxx}-2K\left(\frac{1}{\varepsilon}-1\right)c_4c_{4x}\right\}\left(\frac{y^3}{3}-hy^2\right)$$
$$-\frac{ReNe}{18\ C}c_1c_{1x}\left\{\frac{(y+c_2)^4}{4}-\frac{1}{4}c_2^4-c_2^3y-\frac{3}{2}(h+c_2)^2y^2\right\}$$
$$-\frac{ReNe}{4\ C}c_1^2c_{2x}\left(\frac{y^3}{3}-hy^2\right)$$
$$+Re\frac{Ne}{C}c_1c_{3x}\left\{\frac{4}{15}(y+c_2)^{5/2}-\frac{4}{15}c_2^{5/2}\right.$$

$$-\frac{2}{3}c_2^{3/2}y-\frac{1}{2}(h+c_2)^{1/2}y^2\Big\}\Big]\Big]_x. \tag{28}$$

Finally substituting the leading-order velocity components of (24) and (25), and (27) and (28) at the first order in ξ into the kinematic boundary condition (16), the evolution equation for the surface deflection h(t, x) accurate to $O(\xi^2)$ is obtained such as

$$\begin{aligned}
&h_t+3h^2h_x+\xi\Big\{\frac{6}{5}\mathrm{Re}h^6h_x-\Big(\cot(\beta)-\frac{1}{3}\mathrm{ReNe}c_1c_2^{1/2}\Big)h_xh^3\\
&+\frac{2}{3}\frac{\xi^2}{\mathrm{Ca}}h_{xxx}h^3-\frac{4}{3}Kc_4c_{4x}\Big(\frac{1}{\varepsilon}-1\Big)h^3\\
&+\frac{1}{18}\mathrm{ReNe}(7c_{1x}c_2^{1/2}h+4c_{1x}c_2^{3/2}+6c_1c_2^{1/2}c_{2x})h^3\\
&+\frac{1}{15}\mathrm{ReNe}c_2^{1/2}c_{3x}[8(h+c_2)^{5/2}-8c_2^{5/2}-20c_2^{3/2}h-15(h+c_2)^{1/2}h^2]\Big\}_x\\
&=0.
\end{aligned} \tag{29}$$

The evolution equation is a parabolic partial differential equation and the terms proportional to Ne describe the effect of space charges, and if there is no space charge in the liquid phase, the result has the same form as the one derived by Kim [1997].

LINEAR STABILITY

Linear stability for liquid layer running down an inclined plane was theoretically initiated and studied by Benjamin [1957], Yih [1963] and Gjevik [1970] without the external effect of electric field. Kim et al. [1992], González et al. [1996] and Kim [1997] analyzed the stability of film flow linearly which was being affected under an applied electric field with no embedded space charges in the liquid.

The aim here is to obtain a general result including the space charge effect on the film flow under an electric field. To conduct a linear stability analysis, (29) has to be perturbed about the steady state solution, i.e., $h(t, x)=1+\bar{h}$ where $\bar{h}$ is a small disturbance of the free surface. And because the values of c_i for i=1, ⋯, 5 are dependent on h(t, x), they also have to be expanded about their steady solutions by setting $c_i=c_{is}+\bar{c}_i$ for i=1, ⋯, 5 (overbar values denote the small perturbations superimposed on the steady state solutions of c_{is}), and then $\bar{c}_i$'s have to be denoted with $\bar{h}$ to secure the handiness of linear stability task. The relations between $\bar{c}_i$'s and $\bar{h}$ are gained from the linearized equations of (23).

The results are expressed in the linearized forms like

$$\begin{aligned}
c_1&=c_{1s}+c_{11}\bar{h},\\
c_2&=c_{2s}+c_{21}\bar{h},\\
c_3&=c_{3s}+c_{31}\bar{h},\\
c_4&=c_{4s}+c_{41}\bar{h},\\
c_5&=c_{5s}+c_{51}\bar{h},
\end{aligned} \tag{30}$$

where c_{is}'s can be numerically calculated from (23) at h=1 and c_{i1}'s are determined by using c_{2s} as

$$\begin{aligned}
&c_{11}=\frac{\mathcal{Q}}{\mathcal{P}}Cc_{2s}^{-1/2},\qquad c_{21}=\frac{\mathcal{Q}}{\mathcal{P}},\qquad c_{31}=\frac{8}{3}\frac{\mathcal{Q}}{\mathcal{P}}Cc_{2s}\\
&c_{41}=-\frac{2\varepsilon Cc_{2s}^{1/2}}{(c_{2s}+1)^{1/2}}\Big\{\frac{1}{2}+\Big(1+\frac{1}{2c_{2s}}\Big)\frac{\mathcal{Q}}{\mathcal{P}}\Big\},\qquad c_{51}=-Hc_{41},
\end{aligned} \tag{31}$$

where $\mathcal{P}$ and $\mathcal{Q}$ stand for

$$\begin{aligned}
&\mathcal{P}=3\varepsilon(2c_{2s}+1)(H-1)+8c_{2s}^{3/2}\{c_{2s}^{1/2}-(c_{2s}+1)^{1/2}\}+2(5c_{2s}+1),\\
&\mathcal{Q}=-3c_{2s}\{\varepsilon(H-2c_{2s}-3)+2(c_{2s}+1)\}.
\end{aligned} \tag{32}$$

Now it is possible to perform a linear stability about the system (29) by assuming the small disturbance $\bar{h}$ has a simple periodic form, i.e., $\bar{h}=\exp\{i\alpha(x-ct)\}$, where $\alpha\geq 0$ is a wavenumber of the disturbance and c is the complex wave speed set by $c=c_R+ic_I$.

After putting this harmonic form into (29) and then setting the real part equal to zero with the imaginary wave speed $c_I=0$, the neutral stability condition can be set up by defining the critical Reynolds number

$$\mathrm{Re}_c=\Big\{\cot(\beta)+\frac{4}{3}c_{4s}c_{41}K\Big(\frac{1}{\varepsilon}-1\Big)+\frac{2}{3}\alpha^2\frac{\xi^2}{\mathrm{Ca}}\Big\}/W, \tag{33}$$

where

$$\begin{aligned}
W=&\frac{6}{5}+\frac{2}{3}c_{2s}^{1/2}\mathrm{Ne}\Big\{(c_{2s}+1)^{1/2}c_{31}\Big(\frac{4}{5}c_{2s}^2+\frac{8}{5}c_{2s}-\frac{7}{10}\Big)-c_{2s}^{1/2}c_{31}\Big(\frac{4}{5}c_{2s}^2+2c_{2s}\Big)\\
&+\Big(\frac{1}{2}c_{1s}c_{21}+\frac{1}{3}c_{11}c_{2s}+\frac{7}{12}c_{11}+\frac{1}{2}c_{1s}\Big)\Big\}.
\end{aligned} \tag{34}$$

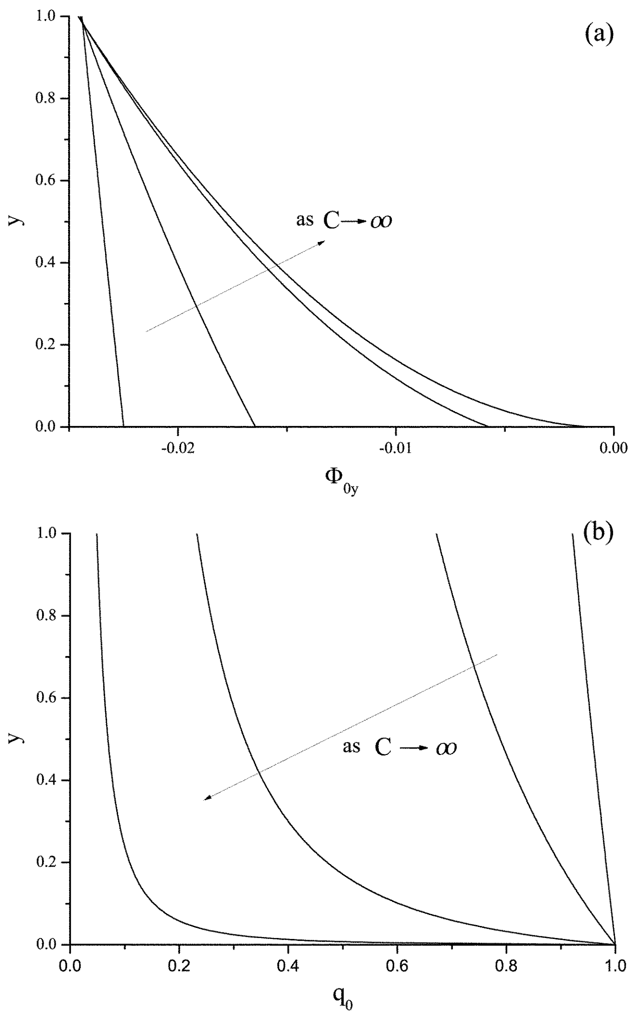


Fig. 2. (a) Leading-order potential gradient profiles in the liquid for $C=0.002\times 5^n$, n=0, ⋯, 3 with ε=10, h=1 and H=5. (b) Leading-order space charge profiles for $C=0.002\times 5^n$, n=0, ⋯, 3 with ε=10, h=1 and H=5.

In the equations of (31) and (32), it has to be noted the value c_{2s} is positive and usually gets smaller than one provided C becomes larger because as the injection is getting stronger the electric field decreases at the injector and, thus from the first equation of Φ_0 in (22), the contribution of c_2 loses its weight. When $C\rightarrow\infty$, it is referred to as a case of space-charge-limited current (SCLC) [Schneider et al., 1970; Castellanos et al., 1992; Atten, 1996]. To confirm the validity of SCLC, the steady-state solutions of Φ_{0y} and q_0 are plotted in Fig. 2 with C increasing for ε=10 and V=10^2 kV/m. As $C\rightarrow\infty$, it can be acknowledged that $\Phi_{0y}\rightarrow 0$, that is, the electric field strength ($=-\Phi_{0y}$)$\rightarrow 0$ and $q_0\rightarrow\infty$ at y=0. The SCLC state is also well represented if substituting c_2=0 into Φ_0 and q_0 in (22).

FINITE-AMPLITUDE SURFACE WAVES

The linear stability analysis is only valid as long as the disturbed surface waves are kept very small enough to have a single harmonic mode. However, beyond the neutral curve defined by (33), that is, in the linearly unstable region around the (α-Re) domain, it has been well known the flow becomes supercritically stable and there exist nearly sinusoidal surface waves of finite amplitude [Gjevik, 1970; Nakaya, 1975]. To investigate this behavior of finite-amplitude surface waves developing on the thin film flow subjected to a unipolar-ion injection, the amplitude and wave-speed equations are derived by letting the surface deflection h(t, x) as a Fourier series,

$$\begin{aligned} h=&1+a\cos(\alpha\varphi)+a^2(b_{2,2}e^{2i\alpha\varphi}+b_{2,0}+b^*_{2,2}e^{-2i\alpha\varphi})\\ &+a^3(b_{3,3}e^{3i\alpha\varphi}+b_{3,1}e^{i\alpha\varphi}+b^*_{3,1}e^{-i\alpha\varphi}+b^*_{3,3}e^{-3i\alpha\varphi})+\cdots. \end{aligned} \tag{35}$$

Here $b_{m,n}$ are arbitrary constants and the asterisk means the complex conjugate, and the disturbance amplitude a(<1) and the phase φ are assumed as [Nakaya, 1975]

$$\begin{aligned} &\frac{da}{dt}=s_1a+s_3a^3+\cdots,\\ &\frac{\partial\varphi}{\partial x}=1,\quad \frac{\partial\varphi}{\partial t}=w=w_1+w_3a^2+\cdots, \end{aligned} \tag{36}$$

where s_1, s_3 which is often called Landau's second coefficient, w_1, and w_3 are some constants and these can be determined with $b_{m,n}$ at the same time.

After substituting (35) into the linearized form of (29) around h=1 and then using the relationships of (36), the film-depth evolution equation can be rearranged according to the order of a. From the equation for the first order of a, s_1 and w_1 are determined from the coefficients of $\cos(\alpha\varphi)$ and $\sin(\alpha\varphi)$, respectively, as

$$s_1(\alpha,\mathrm{Re})=\alpha^2\xi(\mathrm{Re}\mathcal{A}-\mathcal{B}),\quad w_1=-3, \tag{37}$$

where $\mathcal{A}$ and $\mathcal{B}$ represent

$$\begin{aligned} \mathcal{A}=&c_{1s}\frac{Ne}{C}\left\{-\frac{4}{15}c_{2s}^{5/2}c_{31}+(c_{2s}+1)^{1/2}c_{31}\left(\frac{4}{15}c_{2s}^2+\frac{8}{15}c_{2s}-\frac{7}{30}\right)\right.\\ &\left.-\frac{2}{3}c_{2s}^{3/2}c_{31}+\frac{1}{6}c_{1s}c_{21}+\frac{1}{9}c_{11}c_{2s}+\frac{7}{36}c_{11}+\frac{1}{6}c_{1s}\right\}+\frac{6}{5},\\ \mathcal{B}=&\frac{4}{3}K\left(\frac{1}{\varepsilon}-1\right)c_{4s}c_{41}+\frac{2}{3}\alpha^2\frac{\xi^2}{Ca}+\cot(\beta). \end{aligned} \tag{38}$$

Here it has to be noted that if s_1=0, (37) gives the same result as (33) when C in (38) is replaced with the first relation in (23). That is, the result from the first-order equation in a shows the linear stability condition for a monochromatic wave.

The coefficients in a^2 give

$$b_{2,0}=0,\qquad b_{2,2}=-\frac{\mathrm{Re}\mathcal{U}-\mathcal{V}}{\mathrm{Re}\mathcal{M}-\mathcal{N}}+i\frac{135}{\alpha\xi(\mathrm{Re}\mathcal{M}-\mathcal{N})}, \tag{39}$$

where $\mathcal{M}$, $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{V}$ are defined as

$$\begin{aligned} \mathcal{M}=&\frac{Ne}{C}\{6c_{1s}(c_{2s}+1)^{1/2}c_{31}(8c_{2s}^2+16c_{2s}-7)-24c_{1s}c_{2s}^{3/2}c_{31}(2c_{2s}+5)\\ &+5c_{1s}(6c_{1s}c_{21}+4c_{11}c_{2s}+7c_{11}+6c_{1s})\}+216,\\ \mathcal{N}=&240K\left(\frac{1}{\varepsilon}-1\right)c_{4s}c_{41}+840\alpha^2\frac{\xi^2}{Ca}+180\cot(\beta),\\ \mathcal{U}=&\frac{Ne}{C}\left[(c_{2s}+1)^{-1/2}c_{31}\left\{15c_{1s}c_{21}\left(2c_{2s}^2+4c_{2s}+\frac{5}{4}\right)+6c_{2s}^2(6c_{11}+5c_{1s})\right.\right.\\ &\left.+\frac{3}{2}c_{2s}(9c_{11}+10c_{1s})+12c_{11}c_{2s}^3-\frac{21}{2}c_{11}-\frac{105}{4}c_{1s}\right\}\\ &-c_{2s}^{1/2}c_{31}\{15c_{1s}c_{21}(2c_{2s}+3)+6c_{2s}(2c_{11}c_{2s}+5c_{11}+5c_{1s})\}\\ &+\frac{5}{4}\{4c_{11}c_{2s}(3c_{1s}+c_{11})+7c_{11}^2+40c_{1s}c_{11}+18c_{1s}^2\}\\ &\left.+\frac{5}{2}c_{1s}c_{21}(9c_{1s}+8c_{11})\right]+324,\\ \mathcal{V}=&60K\left(\frac{1}{\varepsilon}-1\right)c_{41}(3c_{4s}+c_{41})+90\alpha^2\frac{\xi^2}{Ca}+135\cot(\beta). \end{aligned} \tag{40}$$

In addition, from the analysis of second harmonic component of $\exp(2i\alpha x)$, $b_{2,2}$ has to meet the condition

$$s_1(2\alpha,\mathrm{Re})<0 \tag{41}$$

for its convergence [Nakaya, 1975]. Within the region satisfying $s_1(\alpha,\mathrm{Re})>0$ and (41) the flow system becomes supercritically stable. Hence, there will exist some finite-amplitude surface waves.

Finally, to find out s_3 and w_3, letting the coefficients of $\cos(\alpha\varphi)$ and $\sin(\alpha\varphi)$ in a^3 equal to zero provides the results:

$$\begin{aligned} s_3=&\frac{180}{\xi(\mathrm{Re}\mathcal{M}-\mathcal{N})}+\alpha^2\xi\left(\frac{\mathrm{Re}\mathcal{U}-\mathcal{V}}{\mathrm{Re}\mathcal{M}-\mathcal{N}}\mathcal{C}+\mathcal{D}\right)\\ &+\alpha^2\xi\mathrm{Re}\left\{\frac{\mathrm{Re}\mathcal{U}-\mathcal{V}}{\mathrm{Re}\mathcal{M}-\mathcal{N}}\left(\mathcal{E}-\mathcal{G}-\mathcal{I}-\frac{36}{5}\right)-\left(\mathcal{F}-\mathcal{H}-\mathcal{J}-\frac{9}{2}\right)\right\},\\ w_3=&-\frac{3}{4}+6\frac{\mathrm{Re}\mathcal{U}-\mathcal{V}}{\mathrm{Re}\mathcal{M}-\mathcal{N}}-\frac{135}{\mathrm{Re}\mathcal{M}-\mathcal{N}}\left(\mathcal{C}-\frac{1}{45}\mathrm{Re}\mathcal{U}\right), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \mathcal{C}=&\frac{4}{3}K\left(\frac{1}{\varepsilon}-1\right)(c_{41}^2+3c_{4s}c_{41})+14\alpha^2\frac{\xi^2}{Ca}+3\cot(\beta),\\ \mathcal{D}=&K\left(\frac{1}{\varepsilon}-1\right)(c_{41}^2+c_{4s}c_{41})+\frac{\alpha^2}{2}\frac{\xi^2}{Ca}+\frac{3}{4}\cot(\beta),\\ \mathcal{E}=&\frac{c_{31}}{c_{2s}^{1/2}(c_{2s}+1)^2}\frac{Ne}{C}\left\{c_{1s}c_{2s}c_{21}\left(\frac{2}{3}c_{2s}^3+\frac{7}{3}c_{2s}^2+\frac{8}{3}c_{2s}+1\right)\right.\\ &\left.+c_{11}c_{2s}^2\left(\frac{4}{15}c_{2s}^3+\frac{6}{5}c_{2s}^2+\frac{8}{5}c_{2s}+\frac{2}{3}\right)+\frac{2}{3}c_{1s}c_{2s}^2(c_{2s}+1)^2\right\},\\ \mathcal{F}=&\frac{c_{31}}{c_{2s}^{1/2}(c_{2s}+1)^2}\frac{Ne}{C}\left\{\frac{1}{6}c_{11}c_{2s}^4c_{21}+\left(\frac{1}{4}c_{1s}+\frac{7}{12}c_{11}\right)c_{2s}^3c_{21}\right.\\ &+\left(\frac{1}{2}c_{1s}+\frac{2}{3}c_{11}\right)c_{2s}^2c_{21}+\frac{1}{4}(c_{1s}+c_{11})c_{2s}c_{21} \end{aligned}$$

$$+c_{1s}c_{21}^2\left(\frac{1}{8}c_{2s}^3+\frac{15}{48}c_{2s}^2+\frac{1}{4}c_{2s}+\frac{1}{16}\right)+\frac{1}{6}c_{11}c_{2s}^2(c_{2s}+1)^2\Bigg\},$$

$$\mathcal{G}=\frac{c_{31}}{(c_{2s}+1)^{3/2}}\frac{Ne}{C}\Bigg\{c_{1s}c_{21}\left(\frac{2}{3}c_{2s}^3+2c_{2s}^2+\frac{7}{4}c_{2s}+\frac{5}{12}\right)$$
$$+\frac{4}{15}c_{11}c_{2s}^4+\left(\frac{2}{3}c_{1s}+\frac{2}{5}c_{11}\right)c_{2s}^3+\left(c_{1s}+\frac{11}{10}c_{11}\right)c_{2s}^2$$
$$+\left(-\frac{1}{4}c_{1s}+\frac{1}{15}c_{11}\right)c_{2s}-\left(\frac{7}{12}c_{1s}+\frac{7}{30}c_{11}\right)\Bigg\},$$

$$\mathcal{H}=\frac{c_{31}}{(c_{2s}+1)^{3/2}}\frac{Ne}{C}\Bigg\{c_{1s}c_{21}^2\left(\frac{1}{8}c_{2s}^2+\frac{1}{4}c_{2s}+\frac{9}{64}\right)+\frac{1}{6}c_{11}c_{2s}^3c_{21}$$
$$+\frac{1}{4}c_{2s}^2c_{21}(c_{1s}+2c_{11})+c_{2s}c_{21}\left(\frac{3}{8}c_{1s}+\frac{7}{16}c_{11}\right)+\frac{5}{48}c_{11}c_{21}+\frac{5}{32}c_{1s}c_{21}$$
$$+\frac{1}{6}c_{11}c_{2s}^3+\frac{1}{4}c_{11}c_{2s}^2-\frac{1}{16}(2c_{1s}+c_{11})c_{2s}-\left(\frac{7}{64}c_{1s}+\frac{7}{48}c_{11}\right)\Bigg\},$$

$$I=\frac{c_{31}}{(c_{2s}+1)^2}\frac{Ne}{C}\Bigg\{c_{1s}c_{21}\left(\frac{1}{2}c_{1s}+\frac{4}{9}c_{11}\right)(c_{2s}+1)^2+\frac{1}{9}(3c_{1s}c_{11}+c_{11}^2)c_{2s}^3$$
$$+\left(\frac{5}{12}c_{11}^2+\frac{16}{9}c_{1s}c_{11}+\frac{1}{2}c_{1s}^2\right)c_{2s}^2+\left(\frac{1}{2}c_{11}^2+\frac{23}{9}c_{1s}c_{11}+c_{1s}^2\right)c_{2s}$$
$$+\left(\frac{7}{36}c_{11}^2+\frac{10}{9}c_{1s}c_{11}+\frac{1}{2}c_{1s}^2\right)\Bigg\},$$

$$\mathcal{J}=\frac{1}{(c_{2s}+1)^2}\frac{Ne}{C}\Bigg\{c_{21}\left(\frac{5}{72}c_{11}^2+\frac{1}{3}c_{1s}c_{11}+\frac{1}{8}c_{1s}^2\right)(c_{2s}+1)^2$$
$$+\frac{1}{12}(c_{11}^2+c_{1s}c_{11})c_{2s}^3+\left(\frac{29}{72}c_{11}^2+\frac{17}{24}c_{1s}c_{11}+\frac{1}{8}c_{1s}^2\right)c_{2s}^2$$
$$+\left(\frac{5}{9}c_{11}^2+\frac{7}{6}c_{1s}c_{11}+\frac{1}{4}c_{1s}^2\right)c_{2s}+\left(\frac{17}{72}c_{11}^2+\frac{13}{24}c_{1s}c_{11}+\frac{1}{8}c_{1s}^2\right)\Bigg\}. \tag{43}$$

During the calculation we know the coefficient $b_{3,1}$ is to be zero because otherwise it goes to infinity on the neutral curve $s_1(\alpha, \mathrm{Re})=0$. The coefficient $b_{3,3}$ can be determined from the coefficient equations of $\cos(3\alpha\varphi)$ and $\sin(3\alpha\varphi)$, which has not been written here because it is not necessary to find s_1, s_3, w_1 and w_3.

In the supercritically stable flow region, the initially growing surface wave near the neutral curve will arrive at an equilibrium state of finite amplitude if the second Landau constant s_3 is negative in that region. From the first equation in (36), the finite amplitude of the surface wave can be determined as

$$a=\left(-\frac{s_1}{s_3}\right)^{1/2}, \tag{44}$$

and the speed of the finite-amplitude monochromatic wave c_R is equal to

$$c_R=-w=-w_1-w_3a^2. \tag{45}$$

The calculation results of a and c_R in (α, Re) domain are shown in the next section.

CONCLUSIONS

To examine the interaction of an electric field with a dielectric liquid film flow into which unipolar space charges are steadily injected from a bottom support plane, the solution for the surface deflection is derived accurate to the second order of a thin parameter ξ. Using this evolution equation linear and weakly nonlinear stability analyses are performed.

In the linear analysis, as both values of c_{4s} and c_{41} are always negative as seen from (23) and (31) where $\mathcal{Q}/\mathcal{P}$ is negative and its absolute value is very small ($\ll 1$) compared with c_{2s} (this is always true because it only appears in the perturbed terms), the numerator of (33) for the linear stability analysis has smaller value due to the presence of K for $\varepsilon>1$ when it is compared with the nonelectric case, K=0. The critical value of the Reynolds number becomes diminished by this amount (see Kim [1997]). While the effect of the normal stress on the free surface due to the applied electric field appears in the numerator in (33), the effect of space charges in the layer turns up in W. Considering the terms in the curly brace in W, it is known that the last parenthesized term is dominating over other ones because $c_{1s}(>0)$ is much greater than the small perturbations of c_{11}, c_{21} and c_{31}. Thus the value of W is always greater than 6/5 and this causes the reduction of the critical Reynolds number further more.

Just beyond the neutral stability curve defined by (33) or $s_1(\alpha, \mathrm{Re})=0$, it has been confirmed there exist finite-amplitude surface

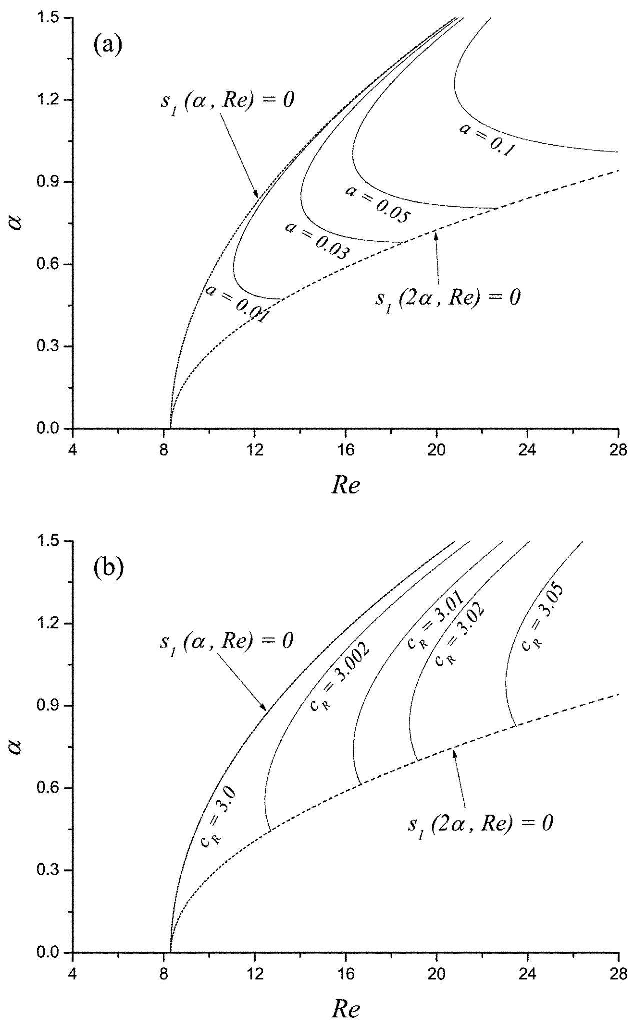
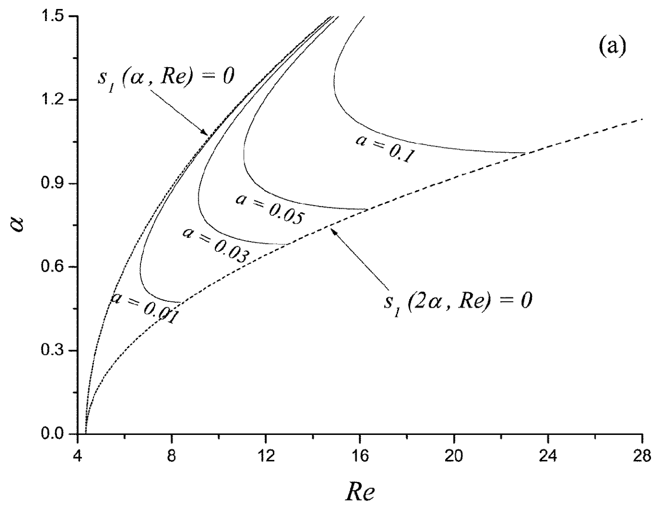


Fig. 3. (a) Amplitude a of periodic wave in the (α-Re) domain when C=0 with K=0 for β=0.1 rad, d=5×10^{-3} m, ξ=0.01 and Ca=1×10^{-5}. (b) Velocity c_R of periodic wave in the (α-Re) domain when C=0 with K=0 for β=0.1 rad, d=5×10^{-3} m, ξ=0.01 and Ca=1×10^{-5}.

waves developing from linearly unstable modes because s_3 is negative ranging from the neutral curve to the limiting curve $s_1(2\alpha, Re)=0$. Hence,the weakly nonlinear analysis has provided formulations to calculate the amplitude and wave speed of any surface wave in the flow region. To compare the effect of space charge on the finite-amplitude waves, (44) and (45) are plotted for the cases of C=0 with K=0 and C=0.1 with K=221.35 ($V=10^2$ kV/m), respectively, by setting β=0.1 rad, $d=5\times10^{-3}$ m, ξ=0.01, H=5, ε=10, $Ca=1\times10^{-5}$ and taking the other physical parameters for water at 0 °C. When C=0 and K=0, i.e., there is no space charge in the liquid as well as no electrical potential, some amplitudes and wave speeds are plotted in (α, Re) domain as Fig. 3a and Fig. 3b, respectively. Here the cutoff Reynolds number Re_c is corresponding to $5/6\cot(\beta)+5/9\alpha^2\xi^2/Ca$. Along a constant wavenumber it is noticed both of amplitude and wave velocity become larger as the Reynolds number increases (Fig. 3a and 3b), which denotes the flow could be easily unstable to a larger Reynolds number. As the space-charged film flow is affected by an electrostatic force, one can expect the system would be much more unstable as shown in Fig. 4a and Fig. 4b where C=0.1 and K=221.35. The shapes of the amplitude and wave speed with the same values as Fig. 3a and 3b are almost similar to the nonelectric case except all of the curves are shifted to the lower region of Reynolds number. That is, in a smaller Reynolds number than the non-electric one the flow gets unstable along the same wavenumber. From this result the weakly nonlinear analysis indicates the presences of space charge makes the value of the Reynolds number smaller than the one in the absence of space charge. That is, the film has more enhanced instability due to space charge in the liquid.



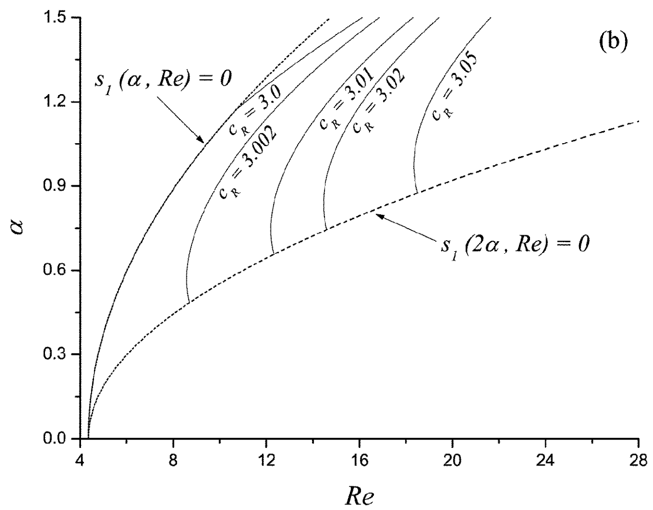


Fig. 4. (a) Amplitude a of periodic wave in the (α-Re) domain when C=0.1 with K=221.35 for β=0.1 rad, ε=10, H=5, $d=5\times10^{-3}$ m, ξ=0.01 and $Ca=1\times10^{-5}$. (b) Velocity c_R of periodic wave in the (α-Re) domain when C=0.1 with K=221.35 for β=0.1 rad, ε=10, H=5, $d=5\times10^{-3}$ m, ξ=0.01 and $Ca=1\times10^{-5}$.

Since the characteristic time scale L/U_0 of the fluid motion is very large when it is compared with the dielectric relaxation time (600 μsec at 0 °C for water) and representative charge migration time (≃4 msec for hydronium ion at 0 °C) based on a field of 10^4 kV/m [Kunhardt et al., 1988], the film flow is governed and limited by its own inertia effect and thus the pseudo-steady-state electrical solutions obtained in this paper are reasonably suitable to the stability analysis in the thin film limit.

It is noted that the major purpose of present study is focused on deriving an film evolution equation describing a thin film flow down an inclined plane subjected to a unipolar-charge injection, and performing linear and weakly nonlinear stability analyses based on the result. Hence the outcomes will be good guides to the experimenters who will need to determine the wave amplitude and wave velocity for their applying fields. Sooner or later the fully nonlinear dynamics related to the present topic will be addressed.

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NOMENCLATURE

a : dimensionless wave amplitude
C : dimensionless number for a relative importance of space charge
Ca : capillary number
c : dimensionless complex wave speed
c_i (i=1, ⋯, 5) : dimensionless coefficients defined in Eq. (23)
d : characteristic film thickness [m]
Fr : Froude number
g : gravity [m/sec^2]
H : dimensionless distance between two electrodes
h : dimensionless free-surface thickness
K : dimensionless electric force constant
K_m : ion mobility [C·sec/kg]
L : characteristic length scale parallel to plane [m]
Ne : electrical Newton number
p : dimensionless pressure
q : dimensionless space charge density
q_i : space charge density at y=0 [C/m^3]
R : dimensionless number for a diffusivity ratio of ions to fluid particles
Re : Reynolds number
s_1, s_3 : dimensionless coefficients for amplitude variance defined in Eq. (36)

t : dimensionless time
U_0 : characteristic velocity [m/sec]
u : dimensionless velocity component of x direction
V : electric potential at y=0 [$m^2 \cdot kg/(C \cdot sec^2)$]
v : dimensionless velocity component of y direction
w_1, w_3: dimensionless coefficients for phase variance defined in Eq. (36)
x : dimensionless distance coordinate parallel to plane
y : dimensionless distance coordinate perpendicular to plane

Greek Letters

α : wavenumber
β : inclination angle of plane with the horizontal
ε_0 : permittivity of free space [8.854×10^{-12} $C^2 \cdot sec^2/(m^3 \cdot kg)$]
ε_f : permittivity of liquid [$C^2 \cdot sec^2/(m^3 \cdot kg)$]
ε : dimensionless permittivity of liquid
μ : viscosity [kg/(m·sec)]
ξ : d/L
ρ : fluid density [kg/m^3]
σ : surface tension [N/m]
Φ : dimensionless electric potential for liquid
φ : wave phase
Ψ : dimensionless electric potential for air

Superscripts

- : small disturbance
* : complex conjugate

Subscripts

0 : leading-order term in ξ
1 : first-order term in ξ
a : air
c : critical value
I : imaginary part
R : real part
s : steady-state value
t : partial derivative with t
x : partial derivative with x
y : partial derivative with y

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