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A Stochastic Analysis of the Flow of Two Immiscible Fluids in Porous Media: The Case When the Viscosities of the Fluids are Equal

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Abstract−A new stochastic theory is developed to explain the flow of two immiscible fluids in porous medium when the viscosity difference between two fluids is zero. In an individual micropore the radius of curvature of the interface separating the fluids is assumed constant and flow is modeled by the random jumping of microscopic interfaces. A one dimensional model composed of an array of parallel capillary tubes of constant radius is analyzed in detail. For the case in which two fluids have equal viscosity an analytical solution is obtained. The fluid displacement process is Fickian and dispersion is described in terms of a diffusion or spreading constant.

Key words: Stochastic Analysis, Immiscible Fluids, Porous Media, Microscopic Interfaces, Fickian Dispersion

INTRODUCTION

Flow in a porous medium continues to attract a great deal of attention both experimentally and theoretically. Original applications from this research were mainly in the field of petroleum recovery [Greenkorn, 1983]. While this application is still important to this day, new emerging applications especially in environmental problems, such as groundwater contamination [Lesage et al., 1992], dispersion of toxic chemicals in soil [Gerstl et al., 1989], hazardous waste transport in deep well disposal [Aubert, 1986], cleaning up of contaminated sites [Noonan et al., 1993] continue to grow. Other application areas [Chun et al., 1990; Kim et al., 1996; Yoon et al., 1998] also get paid increasing attention. Recently, application of this research has been extended to the medical area especially in the fields of cardiology and radiology. Such a wide range of applications should really come as no surprise when one considers the fact that porous structures are ubiquitous in nature.

In general, when two-phase immiscible fluids are flowing in porous media, unstable flow (viscous fingering) may result due to the regime where one fluid is less viscous than the other. The criteria on fluid instability result from viscosity, interfacial tension and properties of media. The displacement of immiscible fluids is of importance in many processes mentioned above. In those areas, an understanding of the relevant mechanisms is essential for the storage and the transport of fluids in porous media. Three different approaches are given to describe the mechanisms linked to the displacement of two immiscible fluids. The first successful theory of fluid instability in case of adverse viscosity ratio was put forward by Saffman and Taylor [1958, 1959] and independently by Chuoke et al. [1959]. This is the most widely studied case as it represents existing conditions in reservoir engineering. The stability theory is analogous to the flow of two immiscible fluids between two closely spaced horizontal plates (Hele-Shaw cell). Secondly, when capillary forces play a dominant role and the viscosities of the two fluids are equal, planar macroscopic interface separating two fluids is found. This arises because capillary forces act equally in all directions resulting in isotropic flow. The concepts of percolation theory were introduced by Broadbent and Hammersley [1957] to study how the random properties of a medium influence the flow of fluid through it.

A new form of percolation theory, called invasion percolation theory, was developed in the 1980's [Chandler et al., 1982; Wilkinson et al., 1983]. Willemsen [1983] also designed a Monte Carlo data processing scheme to analyze the real experiment data. Parallel treatments but with a different interpretation of results were also proposed by De Gennes et al. [1978], Lenormand [1980], Larson et al. [1981]. A major weakness of these percolation models was their exclusive reliance on capillary size. They did not take into account the effects of viscosity and pressure gradient. In a third approach a statistical method, employing the principles of Diffusion Limited Aggregation, was used by Witten and Sander [1981, 1983] to model simple displacement problems. Recently, Paterson [1984] used the DLA approach to simulate two-fluid displacement in porous media when the viscosity of the injected fluid is very small compared to that of the other fluid. Nittman et al. [1985] showed that fractal fingers were obtained when water displaced a polymer solution (of very high viscosity) in a Hele-Shaw cell. The term "fractal" is used to denote any object that has a sprawling, tenuous and fronded pattern. They proceeded to account for their experimental results by using a DLA approach.

Unlike previous approaches, the present analysis is based on a stochastic model. In this model a porous medium is considered to be equivalent to a network of uniform capillaries and flow occurs within a random microscopic capillary network. This paper chiefly concerns on with the dynamic fluid behavior that takes place in a porous medium. An analytical expression is developed and can describe fluid saturation distribution that agrees qualitatively with that seen in experiment [Wang, 1983]. The one-dimensional treatment of the problem is given rigorously and solved exactly for the case when the viscosities are equal. A large lithological and mineralogical variation among different porous media (sandstone, limestone, etc) may affect surface chemistry, resulting in pore-fluid-grain interaction. But this approach can also be extended to systems in which

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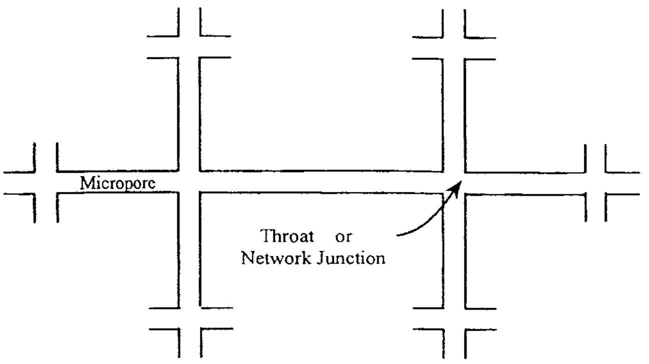

Fig. 1. An ideal network structure of a model porous medium.

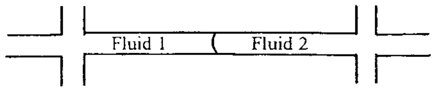

Fig. 2. Microscopic interface with constant radius of curvature in a micropore.

the liquids interact with the porous medium. Due to this advantage the current approach can be used to study various environmentally related problems like the cleaning up of contaminated sites where a thorough knowledge of both the flow pattern of the liquid in the soil and the manner in which they interact in the soil is required.

THEORY

1. A Proposed One-Dimensional Model of a Porous Medium

We may think of a porous medium as a random network structure consisting of pores and throats as shown in Fig. 1. It is assumed that all network elements (micropores) have the same size and all network points (or junctions) have a similar character. In each of the micropores there exists a microscopic interface that separates two immiscible fluids (Fig. 2). Moreover, it is assumed that all these microscopic interfaces have the same radius of curvature and this radius remains virtually constant throughout the displacement. This is a basic postulate of the theory. It is possible that fluctuations in curvature occur and mechanisms such as Haine's jumps, film spreading and contact angle hysteresis do take place in porous medium [Chen, 1986; Chen et al., 1985]. These are neglected in the current model. Also, it has been experimentally shown that the contact angle depends upon the interfacial motion [Dussan, 1976; Ngan et al., 1982]. In this analysis it is assumed that those are minor effects and may be ignored in any macroscopic description of fluid motion in porous media. Because fluids are immiscible, laminar dispersion resulting from the Poiseuille velocity distribution within a given capillary does not occur and plug flow can be assumed everywhere. A stochastic theory can be constructed for fluid displacement such a model media.

In real porous media the distance between particles is of the same order of magnitude as the radius of the particles. Fluids flow through the tortuous path between the particles; these paths moreover vary in a complex and random manner repeatedly joining, dividing and rejoining with each other. Given any two points in a porous medium there are a number of paths connecting them. It is this fundamental character of a porous displacement process that confirms its basic stochastic character. In addition, the size of the particles (and thus the paths) is not uniform; therefore, there is certain randomness associated with fluid motion in a porous medium.

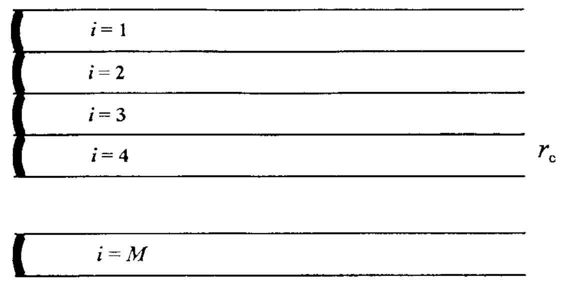

Fig. 3. One dimensional network of a porous medium consisting of M parallel tubes.

Although a real porous medium is three dimensional, the simplest case is for immiscible displacement in a single direction. An array of non-interconnected capillaries is not a satisfactory configuration since it lacks stochastic character. Randomness can be built into the model in the following way. The model of a porous medium still consists of an array of M capillaries as in Fig. 3. Randomness is introduced by considering motion along a capillary as a series of jumping steps rather than a continuous motion. We number the capillaries, i=1, to i=M. It is assumed that the radii of all these capillaries are equal, r_c. We can divide these capillaries into a large number of segments each of length l_p. The length l_p is much smaller than L where L is the total length of a capillary. We also number the segments in each capillary.

A displacement process in this array of tubes is described in terms of the random jumping of an interface from one end of a segment to the other end as shown in Fig. 4. Only one interface in all the capillaries is allowed to jump at any one moment. We are interested in the positions of these interfaces at time scales which are much larger than the time period of a jump.

2. The Case when the Viscosities of the Fluids are Equal

Let's think of two immiscible fluids that migrate in the model of a porous medium. The two fluids are entirely immiscible and have the same viscosity. At t=0, the displaced fluid occupies the complete volume of the capillaries and the interfaces between displaced and displacing fluid are uniformly placed at the entry to each capillary. Between time t=0 and t=t, a pressure difference ΔP is applied to all capillaries and let V cc of displacing fluid enter the array. We write,

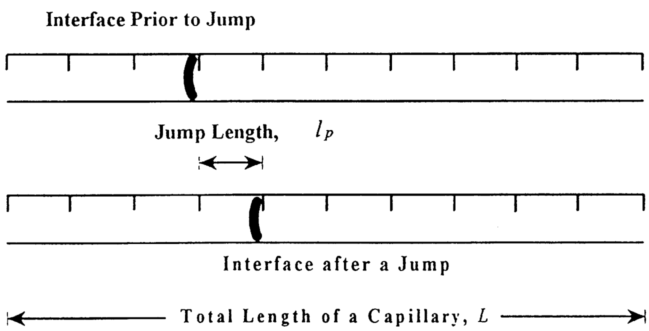

Fig. 4. Basic random fluid step in one dimensional network.

$$V=Qt \tag{1}$$

where Q is the constant volumetric flow rate (ΔP is controlled so that Q is always constant). There is only one interface in each capillary of a porous model during the displacement process. At a certain instant, the probability p, that any one of these interfaces may jump is:

$$p=\frac{1}{M} \tag{2}$$

The probability that an interface will not jump, or will remain at its initial position is

$$q=1-p \tag{3}$$

The total number of jumps taken for a time interval from t=0 to t=t is:

$$N=\frac{V}{\pi r_c^2 l_p}$$
$$N=\frac{Qt}{\pi r_c^2 l_p} \tag{4}$$

Fig. 5 shows the interfaces in all these capillaries may be distributed at the end of this time interval. Contained in the M capillaries there are always a total of M interfaces before breakthrough. We now pose the question: What is the probability that an interface in the first capillary is found at m_1, the interface in the second capillary is found at m_2, the interface in the third capillary is found ... and so on and so forth?

To answer this question we must first consider the probability that the interface in the first capillary is found at m_1 after N jumps irrespective of the positions of the interfaces in the other capillaries. After simple mathematical manipulation the required probability $W(m_1, N)$ can be postulated:

$$W(m_1,N)=\frac{N!}{m_1!(N-m_1)!}p^{m_1}q^{N-m_1} \tag{5}$$

Eq. (5) is based on the assumption that jumps in any capillary at any position or time are equally probable. This is the case only when the fluid viscosities are equal. Under these conditions the pressure drop for a constant flow rate Q will be independent of time. We now ask the question, what is the probability that the interface in the first capillary is found at m_1 and the interface in the second capillary is found at m_2 again after N jumps? From Bayes' formula the resulting equation becomes

$$W(m_1, m_2, N)=W(m_2/m_1, N)\,W(m_1, N) \tag{6}$$

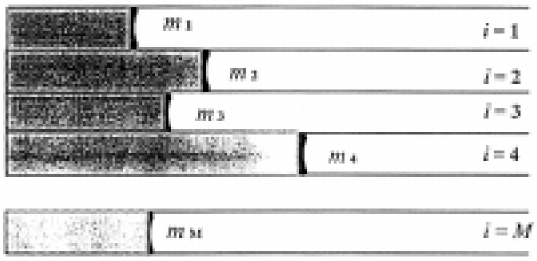

Fig. 5. Random positions of interfaces at some time.

Under aforesaid assumption $W(m_2/m_1, N)$ becomes the probability that the interface in the second capillary is at position m_2, but now after $N-m_1$ steps and in a reduced system containing $M-1$ capillaries. We thus write:

$$W(m_2/m_1,N)\equiv W(m_2,N-m_1)=\frac{(N-m_1)!}{m_2!(N-m_1-m_2)!}p_2^{m_2}q_2^{N-m_1-m_2} \tag{7}$$

$$p_2=\frac{1}{M-1}$$
$$q_2=1-p_2=\frac{M-2}{M-1} \tag{8}$$

Eq. (6) then becomes after substitution from the previous equations.

$$W(m_1,m_2,N)=\frac{N!}{m_1!m_2!(N-m_1-m_2)!}\left(\frac{1}{M}\right)^N (M-2)^{N-m_1-m_2} \tag{9}$$

From the above equations we may conclude that the probability that the interface in the first capillary is at point m_1, the second at m_2, and so on and so forth is:

$$W(m_1,m_2,\ldots m_M)=\frac{N!}{\prod_{i=1}^{i=M} m_i!}\left(\frac{1}{M}\right)^N \tag{10}$$

The above equation follows because the last term in Eq. (9) is unity. If N and all the m's are large numbers ($m_i/N<<1$), we may take logarithms and apply Stirling's approximation to get

$$W(m_1,m_2,\ldots m_M)=A\,\exp-\sum_{i=1}^{i=M-1}\frac{(m_i-\langle m\rangle)^2}{2Np} \tag{11}$$

where the constant A is introduced for proper normalization and $<m>=Np$.

It is convenient to introduce instead of m_i the displacement x_i as:

$$x_i=m_i l_p \tag{12}$$

where l_p is the length of a step. As a result,

$$W(x_1,x_2,\ldots x_M)=\frac{A}{l_p^{M-1}}\exp-\sum_{i=1}^{i=M-1}\frac{(x_i-\langle x\rangle)^2}{2Npl_p^2} \tag{13}$$

Dispersion coefficient D is defined by

$$D=\frac{Qpl_p}{2\pi r_c^2} \tag{14}$$

and the Eq. (13) becomes,

$$W(x_1,x_2,\ldots x_M)=\frac{A}{l_p^{M-1}}\exp-\sum_{i=1}^{i=M-1}\frac{(x_i-\langle x\rangle)^2}{4Dt} \tag{15}$$

Normalization constant A can be estimated since the probability of finding the interfaces all over space is unity. This probability that the interface in the first capillary is at x_1, the interface in the second capillary is at x_2, and so on and so forth, all at time t, i.e. $W(x_1, x_2, \ldots x_M)$ becomes

$$W(x_1,x_2,\ldots x_M)=\left(\frac{1}{4\pi Dt}\right)^{\frac{M-1}{2}}\left(\exp-\sum_{i=1}^{i=M-1}\frac{(x_i+\beta t)^2}{4Dt}\right)\delta(x-x_M) \tag{16}$$

where β the average interface rate is given by:

$$\beta = -\frac{Q}{\pi r_c^2 M} \tag{17}$$

The delta function is defined as:

$$\begin{aligned} &\int \delta(x - x_M)dx = 1 \text{ when } x = x_M \\ &\int \delta(x - x_M)dx = 0 \text{ otherwise} \end{aligned} \tag{18}$$

The reason the delta function arises is because the volumetric flow rate is constant, i.e., at all times

$$\sum_{i}^{M-1} x_i \pi r_c^2 + x_M \pi r_c^2 = Qt \tag{19}$$

The differential equation which satisfies Eq. (16) is the famous diffusion equation and is given by

$$\frac{\partial W}{\partial t} = \beta \sum_{i=1}^{i=M-1} \frac{\partial W}{\partial x_i} + D \sum_{i=1}^{i=M-1} \frac{\partial^2 W}{\partial x_i^2} \tag{20}$$

Thus Eqs. (20) and (16) are the required complete solution for the case when the viscosities of the fluids are the identical.

The probability that the interface in the first capillary is at x_1, the interface in the second capillary is at x_2, ..., the interface in the nth capillary is at x_n, irrespective of the configuration of the remaining M−n interfaces is:

$$W^{(n)}(x_1, x_2, \ldots x_n;\ t) = \iint \ldots \int W(x_1, x_2, \ldots x_M;\ t)dx_{n+1}dx_{n+2}\ldots dx_{M-1} \tag{21}$$

If we define

$$\rho^{(n)}(x_1, x_2, \ldots x_n;\ t) = \frac{M!}{(M-n)!} W^{(n)}(x_1, x_2, \ldots x_n;\ t) \tag{22}$$

then $\rho^{(n)}(x_1, x_2, \ldots x_n; t)$ is the number of interfaces found between x_1 and x_1+dx_1, another between x_2 and x_2+dx_2, and so on and so forth. We also note from Eq. (22) the following normalization condition on $\rho^{(n)}$:

$$\iint \ldots \int \rho^{(n)} dx_1 dx_2 \ldots dx_n = \frac{M!}{(M-n)!} \tag{23}$$

We may use $\rho^{(n)}(x_1, x_2, \ldots x_n; t)$ to find the fluid saturation (for this example fluid is considered the displacing fluid) at x and t. The fluid saturation Φ_W at point x is defined by

$$\Phi_W = \frac{\text{number of capillaries filled with fluid at point x}}{\text{total number of capillaries}} \tag{24}$$

The number of capillaries filled with fluid at point x is equivalent to number of capillaries in which the interfaces are at point x or gone beyond it. We thus write,

$$\Phi_W = \frac{\int_x^\infty \rho^{(1)}(x,t)dx}{M} \tag{25}$$

also

$$-\frac{\partial \Phi_W}{\partial x} = \frac{\rho^{(1)}(x,t)}{M} = P^{(1)}(x,t) \tag{26}$$

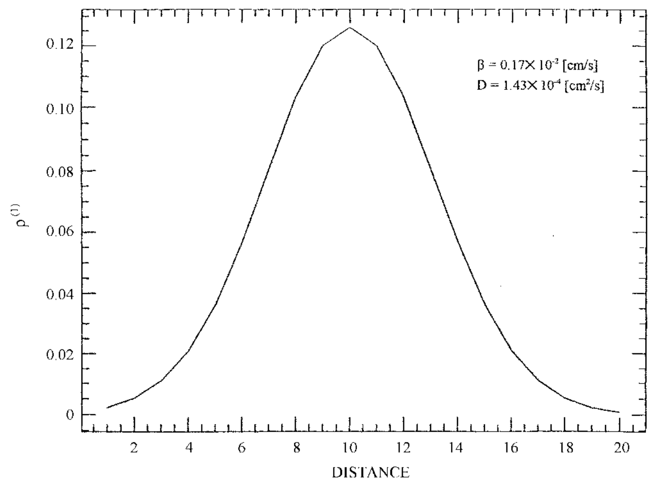

Fig. 6. Probability of finding an interface as a function of position.

where $\rho^{(1)}(x, t)$ is given by

$$\rho^{(1)}(x,t) = M\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} W(x_1, x_2, \ldots x_M)dx_2dx_3\ldots dx_{M-1} \tag{27}$$

$\rho^{(1)}(x, t)$ is most important since it gives the number of interfaces between x and x+dx at t. Substituting Eq. (16) into Eq. (27) and carrying out the integration we will finally obtain:

$$\rho^{(1)}(x,t) = \frac{M}{(4\pi Dt)^{0.5}} \exp -\frac{(x+\beta t)^2}{4Dt} \tag{28}$$

Eq. (28) enables the distribution function $\rho^{(1)}(x, t)$ to be obtained from fluid saturation data. Such saturation data is commonly obtained when medical imaging techniques, such as CT/MRI, are used to study fluid flow in a porous medium.

A graph of Eq. (28) is illustrated in Fig. 6. Substituting the above equation into Eq. (25) we get:

$$\Phi_W = \frac{M}{M} \frac{1}{(4\pi Dt)^{0.5}} \exp -\frac{(x+\beta t)^2}{4Dt} \tag{29}$$

$$= \frac{1}{2}\left\{1 - \operatorname{erf}\left(\frac{x-\beta t}{2\sqrt{Dt}}\right)\right\} \tag{30}$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{x+\beta t}{2\sqrt{Dt}}\right) \tag{31}$$

where the error function is:

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp -y^2 dy \tag{32}$$

A graph of fluid saturation is given in Fig. 7.

CONCLUSIONS

A new stochastic theory has been proposed to describe fluid flow in a porous medium that can be modeled as an interconnecting network of identical pores and throats. Before an investigation of a three- (or two-) dimensional version, a one-dimensional model was analyzed in depth. In the one-dimensional model the porous medium is modeled as an array of parallel capillary tubes all of constant ra-

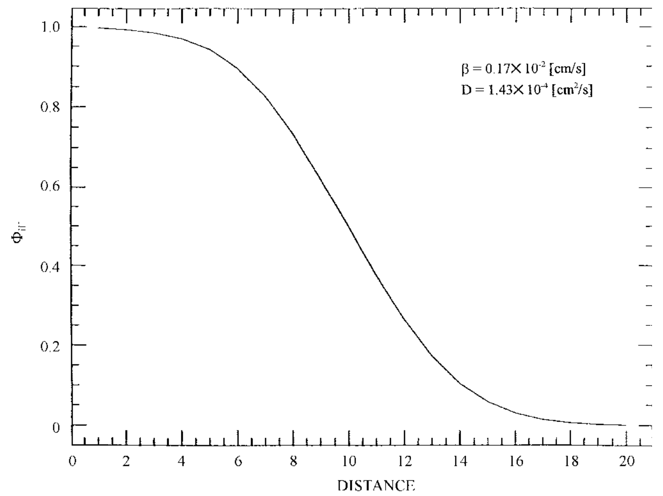

Fig. 7. Displacing fluid saturation as a function of position.

dius. The capillary tube is divided into a large number of segments whose length is much smaller than the total length of the capillary. Fluid displacement is modeled by the random jumping of the microscopic interfaces from one end of the segment to the other end and the jumps are independent. In the case when the viscosities of the two fluids are equal, an analytical solution is given and fluid displacement can be described by the diffusion equation. It is shown that dispersion coefficient can be proportional to the volumetric flow rate and the displacement process can be explained to be Fickian.

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NOMENCLATURE

D : dispersion coefficient [m^2/s]
L : length of the porous medium [m]
l_p : length of a jump [m]
M : total number of capillaries [number]
<m> : average number of jumps taken by an inter face [number]
m_i : number of jumps taken by the interface in capillary i [number]
N : total number of jumps taken from time t=0 to t=t [number]
P : pressure [Pa]
p : probability that any one interface may jump [-]
Q : volumetric flow rate [m^3/s]
q : probability that any one interface may not jump [-]
r_c : radius of a capillary [m]
t : time [s]
V : volume of displacing fluid [m^3]
W(m_1, N) : the probability that the interface in capillary 1 takes m_1 jumps when the total number of jumps taken by all interfaces (including capillary 1) is N [-]
W(m_1, m_2, ... m_M) : the probability that the interface in capillary 1 takes m_1 jumps, the probability that the interface in capillary 2 takes m_2 jumps, and so on and so forth, when the total number of jumps taken by all interface is N [-]
W(x_1, x_2, ... x_M; t) : probability that the interface in the first capillary is found between x_1 and x_1+dx_1, the interface in the second capillary is found between x_2 and x_2+dx_2, and so on and so forth, all at time t [$1/m^{n-1}$]
$W^{(n)}$(x_1, x_2, ... x_M; t): probability that the interface in the first capillary is found between x_1 and x_1+dx_1, the interface in the second capillary is found between x_2 and x_2+dx_2, and so on and so forth, the in- terface in the n-th capillary is found between x_n and x_n+dx_n, irrespective of the positions of the other interfaces, at time t [$1/m^n$]
x_i : position of interface i [m]

Greek Letters

β : average displacement velocity at which all the interfaces move [m/s]
ϕ : porosity of the porous medium [-]
Φ_w : fluid saturation [-]
μ : viscosity of displacing fluid [kg/ms]
$\rho^{(1)}$(x, t) : the number of interface between x and x+dx at time t [number/m]

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